Now it is obvious that *before the idea of "mass" can be used in the equations of dynamics, the "masses" of bodies must be defined as measurable quantities;* and all the different definitions of the word mass are in reality different definitions of the way in which the measurement of mass is supposed to be effected.

III. Current Definitions of Mass.

We now take up, in order, the four principal types of definitions of mass contained in the current textbooks. The books referred to are listed at the end of the article.

1. *Mass as Measured by Force per Acceleration (Inertia).*

In the opinion of J. Clerk Maxwell (p. 40), "the only definition of equal masses which can be admitted in dynamics" is the following: "Any two bodies are of equal mass if equal forces applied to these bodies produce, in equal times, equal changes in velocity"; and this definition is in fact the one adopted by the largest number of authors. Thus:

Lamb (p. 20): "It is observed . . . that the same force applied in succession to different bodies produces in general different degrees of acceleration. This is described as due to differences in the inertia or mass of the respective bodies. Two bodies which acquire equal velocities in equal times, when acted on by the same force, are regarded as dynamically equivalent, and their masses are said to be equal."

Poincaré (p. 13, tr.): "The quotient of force divided by acceleration is what we mean by the mass of a body."

E. H. Hall (p. 142): "Equal masses are, by definition, quantities of matter which, whatever their inequalities in other respects, are alike in this, that they require equal forces to give them equal velocities in equal times."

Similarly in Routh (p. 23), Poynting & Thomson (p. 4), Rankine (p. 482), Loney (p. 5), Cox (p. 118), Clifford (p. 59), Macgregor (p. 191), Garnett (p. 22), Williamson & Tarleton (p. 32), H. Adams (p. 10), Encyclopædia Britannica, Art. Mechanics (p. 974), New International Encyclopædia, Art. Matter (p. 1033), Lanza (p. 11), Trowbridge (p. 61), DuBois (p. 3), Wright (p. 46), Carhart (p. 42), Kimball (p. 18), Duff (p. 28), Slocum (p. 70), Ferry (p. 77), Mann & Twiss (p. 36), Thwing (p. 12), Trautwine (p. 336), Hayward (p. 194), Chwolson (p. 72), Deschanel (p. 44), Delauney (p. 119), Jamin (p. 62), Massau (p. 111), Ringelmann (p. 99), Violle (p. 98), La Grande Encyclopédie, Art. Masse, subheading Mécanique (p. 371), Autenrieth (p. 194), Hamel (p. 46), Streintz (p. 100), Voigt (p. 39), Voss (p. 49), Wüllner (p. 56), "Hütte" (p. 148), Weber & Wellstein (p. 93).

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we hardly say: this color is exactly "twice as red" as that. Such entities may be said to be comparable, but not additive. On the other hand, many sets of entities satisfy *B* but not *A*. For example, two points in *n*-dimensional space have a readily definable "sum," but we do not speak of either of such points as being "less than" the other. Such entities may be said to be additive, but not comparable. No entities can be said to be strictly measurable unless they are both comparable and additive.

Also in the following books, if, as is clearly intended, we understand by "inertia" the ratio of force over acceleration: Tait (p. 28), Gray (p. 109), R. W. Stewart (p. 93), Morley (p. 27), Martin (p. 4), Slate (p. 224), Fuller & Johnston (p. 5), Culler (pp. 4, 20), Ames (1897, p. 31), Crew & Jones (p. 25), Carhart & Chute (p. 6), Duhem (p. 105).

This definition of mass as force ÷ acceleration is obviously unintelligible without some previous definition of the equality of two forces, and this is expressly pointed out by a number of writers. Thus:

Loney (p. 5): "It is here assumed that it is possible to create forces of equal intensity on different occasions, *e. g.*, that the force necessary to keep a given spiral spring stretched through the same distance is always the same when other conditions are unaltered."

R. W. Stewart (p. 92): "Equal forces may be defined as forces which extend the same spiral spring to the same extent."

Similarly, Voss (p. 51), Cox (p. 118), Chwolson (p. 70), Larmor (p. 278), and others.

Or, if preferred, equal forces may be defined as forces which give to the same lump of metal equal accelerations. In any case, it must be remembered that forces can be specified and compared without involving in any way the idea of the mass of a body.[1]

(1*a*) A variation of this definition (by force over acceleration) is adopted by some writers who replace the general expression F/a by W/g, where W is the special force due to gravity, and g the corresponding acceleration. (Older writers use G/g for W/g.)

Thus I. P. Church (p. 53): "Since [for any given body] the quotient G/g is invariable, it will be used as the measure of the mass M in the body."

C. M. Woodward (p. 176): "The ratio W/g is quite universally represented by one letter m, which stands numerically for the mass of the body."

Similarly, Winkelmann (p. 4), Merriman (p. 148), Moulan (p. 129), Kent (p. 488), Larousse (p. 1310), Wood (p. 16), Worthington (p. 9), Perry (Calc., p. 26, Mech., p. 40), and probably Weisbach (p. 158), Hancock (p. 4), and Hudson (p. 74).

(1*b*) Another variation, expressed in terms of kinetic energy, is found in J. J. Thomson (pp. 28–30), Larmor (p. 181), Sanford (p. 27), Cheston, Gibson & Timmerman (p. 94), and Young (p. 108). Compare Holman, p. 61.

2. *Mass as Measured by a Beam Balance (Standard Weight).*

In spite of Maxwell's opinion that force divided by acceleration is the only admissible definition of mass, many writers in both pure and applied mechanics prefer a second definition, which we shall call the beam balance definition. Thus:

Franklin & MacNutt (pp. 25, 14): "The mass of a body, as a quantity, is

[1] Daniell's contention (p. 20) that "Force can never be measured until we know . . . the mass acted upon and the acceleration actually imparted to it" would appear to be refuted by the whole history of the science of statics.

defined by the operation of weighing by a balance. . . . The only proper definition of a quantity is the definition which corresponds to the fundamental method which is actually used in measuring that quantity. . . . It is all very well to *talk* about defining the mass of a body in accordance with the above utterly impracticable method of measuring its mass [by force and acceleration]; but sensible men always *define* things in physics in the way they *do* them."

Ganot (p. 15): "Two bodies are said to have equal masses when, if placed in a perfect balance in vacuo, they counterpoise each other."

Pender (p. 930): "Two bodies are said to have equal masses if, when they are suspended simultaneously in a vacuum, one from each end of an equal-armed balance, there is no tipping of the beam."

Capito (p. 170): "This is the only way in which the mass of a body can be found direct."

Similarly, in Love (p. 68), Maurer (p. 142), Stewart & Gee (p. 63), Daniell (p. 12), Nichols & Franklin (p. 8), Gage (p. 2), Henderson & Woodhull (p. 39), Hoadley (p. 15), Müller-Pouillets (p. 94).

Also in the following books, where "weighing" is clearly to be understood as "weighing on a beam balance": Thomson & Tait (p. 220), Jeans (p. 29), Lorentz (p. 118), Planck (p. 2), Minchin (p. 2), Newcomb (p. 554), Wentworth & Hill (p. 7), Dolbear (p. 19), C. F. Adams (p. 10), Everett (p. 16), Ziwet & Field (p. 120), Hastings & Beach (p. 5); and probably Miller & Lilly (p. 2), Black & Davis (pp. 154–155), and Föppl (p. 30).

The essential equivalence between the force ÷ acceleration definition and the beam balance definition is maintained by many writers. Thus

Kennelly (p. 8): "Mass [of a body] is estimated either by its inertia or by its weight."

Rowland & Ames (p. 46): "It is possible to define two objects as having equal masses if, when set in motion by the same cause, they have identical motions. [This is the force ÷ acceleration definition.] . . . Or, two bodies might be defined as being of equal mass if they have the same weight. [This is the beam balance definition.] . . . There is no a priori reason why there should be any connection between these two definitions; but it is found by experiment . . . that two bodies which have the same inertia also have the same weight. Consequently, it is immaterial which of these two properties is taken as the basis of comparison. . . . In practice masses are compared and measured by means of a balance, which is an instrument to measure weight."

It should not be forgotten, however, that the equivalence between these two definitions is merely an assumption, and indeed one which seems likely to be called into question by modern researches into the behavior of bodies moving with very high velocities. For a further discussion of the contrast between these two definitions, see § VI, *F*, below.

3. *Mass as Measured by Mutual Acceleration (Interaction).*

The following definition was first proposed by Mach in 1868,[1] and is associated in England with the name of Karl Pearson.

Mach (p. 218): "Those bodies are bodies of equal mass which, mutually acting on each other, produce in each other equal and opposite accelerations."

Pearson (p. 329): "We conceive a standard corpuscle, Q; . . . then

$$\text{Mass of } A = \frac{\text{Acceleration of } Q \text{ due to } A}{\text{Acceleration of } A \text{ due to } Q}.$$

We have here a perfectly clear and intelligible definition of the mass of A relative to Q. It is in this manner that mass is invariably determined scientifically."

Appell (p. 89): "The ratio of the masses of two points is, by definition, the inverse of the ratio of the accelerations which each of them determines in the other; the numerical value of one mass having been chosen arbitrarily, the values of all the others are determined. The word force does not enter into the principles of dynamics as here developed."

Hoskins (p. 227): "The masses of two particles are in the inverse ratio of the accelerations which they give each other. This is a definition of mass."

Similarly, in Goodwill (p. 48), Encyclopædia Britannica, Art. Motion (p. 907), Barton (p. 196), Roberts (p. 72), Crew (Mech., p. 66, Physics, pp. 57–59), Ames (1904, p. 60), Lecornu (p. 207), Andrade (p. 54).

Also, as an alternative definition, in Auerbach (p. 36), and others.

The most obvious case of interaction between two bodies is the case of impact or collision, and hence most of the writers who use the mutual acceleration definition regard a "*collision balance*," that is, an apparatus for comparing velocities before and after collision, as the fundamental instrument for measuring masses. See, for example, Goodwill, pp. 50–53.

As to the relation between this definition and the preceding definitions: Love (p. 168) makes a distinction between the "mass-ratio" of two bodies, as determined by the mutual acceleration definition, and the "ratio of the masses" of the bodies, as determined by the beam balance definition, and asserts the equality of the two. Pearson (pp. 333–337) shows how, on certain assumptions, the mutual acceleration definition may be regarded as including both the beam balance definition and the force-per-acceleration definition as special cases. (Compare Love, pp. 168, 340.) On the other hand, if we accept the principle of action and reaction, the collision balance definition is obviously a special case of the force-per-acceleration definition, since the bodies are acted upon, during the brief interval of impact, by equal forces.

(3*a*) The purely analytic definitions of mass given by Kirchhoff (p. 23), Boltzmann (p. 21), Timerding (p. 287), and Webster (p. 23) may also be classed under this heading (of mutual acceleration).

[1] E. Mach, "Ueber die Definition der Masse," *Carl's Repertorium für Experimental-Physik*, Vol. 4, pp. 355–359 (1868).

These three types of definitions are the ones which the student is most likely to meet in his ordinary reading. (Compare The Century Dictionary, Art. Mass.)

4. *Mass as Measured by the Number of Identical Corpuscles in a Body.*

The following definition depends on a certain assumption in regard to the structure of matter.

Johnson's Cyclopædia, Art. Dynamics (p. 875): "If we suppose . . . that the ultimate particles or molecules of all substances are the same, and that we may designate by the term density the degree of proximity of the particles of any body to each other, then the number of particles in a given volume may be taken to denote the mass of the body."

Hertz (p. 46): "The number of material particles in any space, compared with the number of material particles in some chosen space at a fixed time, is called the mass contained in the first space."

La Grande Encyclopédie, Art. Masse, subheading Astronomie (p. 371, tr.): "One can then define the mass of a body as the number of identical particles (points matériels) of which it is composed."

Houston & Seal (p. 30): "The mass of a body is proportional to the number of its molecules."

This definition of mass, though seldom explicitly stated, is probably the idea that most authors have in mind when they speak vaguely of mass as "quantity of matter."

Granting the basic assumption, the definition is logically quite defensible. Unfortunately, however, there is not a shred of evidence in support of this assumption, and the student who has been brought up on any such notion of the ultimate structure of matter finds it difficult to adjust himself to the modern theories, which make the structure of the atom as complex as that of the solar system. And of course, as a practical method of determining the mass number of any given body, the definition fails completely, and resort must be had, sooner or later, to one of the other methods of measuring mass. (Compare Barton, p. 195.)

IV. Incomplete Definitions of Mass.

All the definitions so far mentioned satisfy the fundamental condition that masses must be measurable quantities. But a number of widely used text-books still retain "definitions" of mass which are no definitions at all, because they fail to explain how mass is to be measured.

5. "*Quantity of Matter.*"

Almost every book on the subject pays its respects, in some form or other, to the time-honored "definition" of mass as "quantity of matter."

In compiling the present report, whenever I found this "definition" supplemented by a statement of how the "quantity of matter" is to be measured, I

ignored the words "quantity of matter" and accepted this supplementary statement as the author's real definition of mass. But where the words "quantity of matter" stand alone (as in Ziwet, p. 129), they must be regarded as a totally inadequate definition of the mass of a body as a quantitative concept. The situation is well described by many authors. For example:

Encyclopædia Britannica, Art. Motion (p. 907): "The mass of a body is often loosely defined as the measure of the quantity of matter in it. This definition correctly indicates that the mass of any portion of matter is equal to the sum of the masses of its parts, . . . but gives no test for comparison of the masses of bodies of different substances."

Jamin (p. 63, tr.): "Some authors define mass as the quantity of matter which the body contains. . . . This idea is vague and illusory, for matter is not a thing which one can measure. . . . Those who wish to preserve this idea of mass are obliged, at the start, to define what they propose to mean by equal quantities of matter."

Chwolson (p. 73, tr.): "The primitive definition of the term mass as quantity of matter is not admissible, for . . . for heterogeneous materials, the notion of equal or unequal quantities of matter is entirely lacking."

Hoskins (p. 2): "The mass of a body is often briefly defined as its quantity of matter. These words, however, convey no definite idea of the meaning of mass as a factor in the determination of motion."

It cannot be emphasized too strongly that without a definite statement of the method of measurement, the phrase "quantity of matter" is empty and useless in dynamics. *Merely calling a thing a quantity does not make it a quantity.* It is just as absurd to speak of "quantity" of matter without defining the method of measurement, as it would be to speak of "quantity" of beauty, or "quantity" of temperature, without defining some method of estimating the values of these "quantities."

There are, of course, familiar cases in which we do have a direct intuitive perception of comparative "quantity." For example, any one can tell that one speed is many times faster than another speed, or that one bulk is many times larger than another bulk. Speed and bulk are perfectly definite intuitive magnitudes, regardless of the method of measurement which may be used to determine their numerical value. *But in the case of matter, it is precisely this intuitive perception of quantitative relations which is entirely lacking.*

Suppose, for example, that a load of coal and a load of wood come to the door, and let us ask the question, which has more "matter" in it? It will be found, on reflection, that no answer to this question is forthcoming, *except* by reference to one or other of the measurable properties described above. We can tell at once which load has more bulk; we can hazard a guess as to which has more weight, or as to which would require more force to set it in moton; we can even amuse ourselves by pretending to count the number of identical particles of which we may imagine the two loads to consist; but when we try to think of any direct comparison between the mere "matter" in one load with the mere

"matter" in the other load, without making even subconscious use of any one of these methods of measurement, the mind becomes absolutely blank. No such direct comparison is possible, even in theory (except, of course, in the often cited but quite trivial and irrelevant case of the comparison of bodies of the same homogeneous material).

Fortunately for the student, the number of textbook writers who still persist in defining mass as *merely* "quantity of matter" is rapidly decreasing (since the publication of Pearson's *Grammar of Science*), and is already negligibly small.

6. *Circular Definitions.*

The definition of mass in terms of density, followed by a definition of density in terms of mass, still appears in Tait and Steele (p. 42), Bowser (p. 6), Andrews & Howland (p. 11). (Compare Thomson & Tait, p. 220.) The obvious circularity of this definition has been pointed out by many critics. Similarly, Dadourian (p. 103) defines mass in terms of "kinetic reaction," and on the same page defines "kinetic reaction" in terms of mass.

7. *Definition Lacking.*

A number of books in which one would expect to find a definition of mass use the term freely without formal definition of any kind. For example, Jamieson (p. 2), Millikan (p. 13), Smith & Longley (p. 90), Hedrick & Kellogg (p. 1), Watson (p. 68), Silberstein (p. 50).

(7*a*) On the other hand, some very successful writers prefer to develop the whole theory without using the *word* mass at all; so Cotterill & Slade (*loc. cit.*), Merriman (p. 148), Black & Davis (p. 154), and especially Sir George Greenhill (*loc. cit.*).

8. *Mass as Lump of Matter.*

One further use of the word mass may be mentioned, although it hardly pretends to be a definition of the mass of a body in any quantitative sense. When one speaks of "the impact of two masses," or when one solves a problem about "two masses suspended by a cord over a pulley," one obviously means by *a mass* simply *a body*, or *lump of matter*. This is a very common and convenient usage which ought not to be likely to create any misunderstanding.

This completes the record of all the important uses of the word mass in the current textbooks.

It remains to add a brief report on the various names that are given to the units of mass.

V. Remarks on the Names of the Units of Mass.

(1) If the mass of a body is defined by the force-per-acceleration definition, the name of the unit of mass is naturally derived from the names of the units of force and acceleration, just as the name of the unit of acceleration is itself

derived from the names of the units of length and time. Thus we speak of a mass of 1 lb. per ft.-per-sec.^{2}, or 1 kg. per cm.-per-sec.^{2}.

Worthington (p. 9) abbreviates 1 lb. per ft.-per-sec.^{2} into 1 "slug," while Maurer (p. 143) calls this unit a "gee-pound."

On the other hand, some writers prefer to leave the unit unnamed altogether.

Church (p. 53): "No name will be given to the unit of mass, it being always understood that the fraction G/g will be put for M before any numerical substitution is made."

Similarly, Perry (Calc., p. 26), Sanford (p. 28).

(2) If the beam balance definition or the mutual acceleration definition is adopted, the name of the unit of mass is simply the name of the lump of metal which is used as the standard; as the standard pound body, the standard kilogram body, etc.

It must be remembered, however, that the words pound, kilogram, etc., are used also (quite properly) to denote units of force. Hence, in books in which forces and masses appear in the same equations, it is necessary to distinguish between the *pound force* and the *pound mass*, and between the *kilogram force* and the *kilogram mass*.

This distinction is emphasized by many writers, for example, Kennelly (p. 10). The attempt sometimes made, however, to symbolize this distinction by writing "lb." for one of the units and "pd." for the other seems foredoomed to failure, since no one can remember which is which. For example, Worthington (p. 9) uses "pd." for force and "lb." for mass, while Dadourian (p. 109) uses "pd." for mass and "lb." for force. Similarly, the two new notations recently proposed by Hudson (p. 74)—"pounds (abs.)" for mass as measured by weight, and "pounds (grav.)" for mass as measured by weight divided by g—would seem to serve only to increase the existing confusion.[1]

In books in which forces are compared only with forces, and masses only with masses—so that no equation contains both forces and masses together—the necessity for this distinction disappears. This is the real secret of the simplicity, as regards units, of the method advocated by the present writer (see § VI, *H*, below).

VI. Conclusions.

In conclusion, I venture to express my own personal preference, in regard to the use of the word mass, by offering to defend the following propositions:

(*A*) It is impossible to assume, at the beginning of a course in mechanics,

[1] As an illustration of the confusion which arises when units of force and mass are introduced into the same equation, it may be amusing as well as instructive to reproduce the following statements from two books which have had a wider circulation perhaps than any others in their respective fields of theoretical and applied mechanics.

Routh (*Dynamics of a Particle*, p. 25): "The equation $W = mg$ shows that the weight of a unit mass is g."(!) (One had always supposed that W was a force, and g an acceleration!)

Weisbach (*Mechanics of Engineering*, p. 159): "Hence the mass of a body whose weight is 20 pounds is 0.62 pounds; and inversely the weight of a mass of 20 pounds is 644 pounds."(!)

Quotations almost as distressing as these might be made from many more modern textbooks.

that a student has any intuitive idea of the meaning of the expression "the mass of a body." Before this term is introduced, its meaning should be defined (§ 7). The use of the word "mass" as a synonym for "lump of matter" is often convenient, and need not be discouraged (§ 8); but this gives, of course, no quantitative idea of the "mass of the body."

(*B*) The definition of "the mass of a body" as the number of identical corpuscles in it (§ 4), and the pseudo-definition of "mass" as "quantity of matter" (§ 5), are unsatisfactory, and should be abandoned.

(*C*) The definition of "mass" by mutual acceleration, at least in its general form (§ 3), is too abstract for beginners.[1]

There remain to be considered, therefore, only the inertia definition (§ 1) and the beam balance definition (§ 2).

(*D*) The idea of the *inertia of a body*, as measured by force divided by acceleration, is important, and the Newtonian hypothesis that the inertia of a body is constant (that is, independent of the body's velocity) should be thoroughly understood. Any teacher who wishes to introduce the term "mass of a body" to mean the inertia of the body has excellent authority for doing so (§ 1).

(*E*) The idea of the *standard weight of a body*, as measured by a beam balance, is also important, and any teacher who wishes to introduce the term "mass of a body" to mean the standard weight of the body has excellent authority for doing so (§ 2).

(*F*) *It is by no means a matter of indifference, however, which of these two definitions of "mass" is adopted,* since many passages in standard literature presuppose one of these definitions to the absolute exclusion of the other.

As a first example, consider the following quotation from Poincaré (p. 13, tr.): "In the new mechanics, the mass of a body increases enormously with the velocity, and becomes infinite when the velocity approaches the velocity of light." Here the word "mass" clearly must mean inertia; the passage would become quite unintelligible if "mass" were interpreted as standard weight.—Secondly, consider the following proposition, cited by many authors as a fundamental law of nature: "Force = mass × acceleration." Here the word "mass," whatever else it may mean, certainly does *not* mean inertia (that is, force ÷ acceleration), since, if it did, the proposition in question would be not a law of nature, but merely a trivial algebraic transformation of the very definition of inertia. To give the proposition any significance as a law of nature, the word mass must be interpreted as something other than inertia, for example, as standard weight.—Thirdly, consider the contrast between the two ways of naming the units. If the inertia definition is adopted, the natural units of mass are the lb. per ft.-per-sec.^{2}, the kg. per cm.-per-sec.^{2}, etc. (where "lb." and "kg." are to be understood as units of force); while if the beam balance definition is adopted, the natural units of mass are the pound-mass, the kilogram-mass, etc. (which are the names

[1] The collision balance, which is a very instructive apparatus, is best interpreted as an instrument for comparing the inertias of two bodies, the equality of the forces acting on the two bodies during the impact being insured by the principle of action and reaction (§ 3).

of certain lumps of metal); moreover, the latter units differ from the former by the numerical factors 32.1740 and 980.665 respectively.

These three illustrations may suffice to show that the choice between the two definitions of mass is a serious matter, which affects the whole development of the course.

(*G*) *In view of these considerations, it would appear desirable, at least at the beginning of the course, to employ only the separate, well-established terms "inertia of a body" and "standard weight of a body" to denote these two closely related but still quite distinct conceptions.* Later in the course, the student should be told, as an important matter of general information, that the word "mass" is used by some writers to denote inertia and by other writers to denote standard weight. *There is at any rate nothing to be lost by following this plan; and there is certainly much to be gained in the way of clear thinking.*

All these recommendations, it will be noticed, leave entirely open the question whether $F/a = F'/a'$ or $F = ma$ is the better form of the fundamental equation of mechanics, and I hope that a general acceptance of the truth of the propositions (*A*)–(*G*) may serve to clear the ground for a more satisfactory discussion of that question.

(*H*) To avoid possible misunderstanding, I should like to add that while I have constantly advocated the use of the equation $F/a = F'/a'$ as the fundamental equation of mechanics, I have no objection whatever to the *use of the letter m as an abbreviation for F/a, or W/g.* It is obvious, however, that the letter m, when so used, denotes an inertia, that is, a force-per-acceleration, and not a number derived from a beam balance. Hence, when numerical computation is involved, I like to replace such an m by the more explicit W/g before inserting numerical values, thus avoiding the possibility of confusion between the units of inertia and the units of standard weight. By following this plan, nothing is lost in the way of algebraic compactness, and much is gained in the way of sureness and comfort in numerical computation.[1]—The alternative plan of regarding the "m" in $F = ma$ as a fundamental concept (usually vaguely defined as "quantity of matter") appears to me to offer no advantages, and to lead, in practice, to many disturbing and quite superfluous complications.

VII. List of Books Examined.

This list includes merely those books on mechanics or physics which happened to come readily to hand, and is not intended to be in any sense exhaustive. The numbers in [ ] refer to sections of the present paper.

C. F. Adams, *Physics for Secondary Schools*, Amer. Book Co., 1908 [2].
H. Adams, *Cassell's Engineers' Handbook*, Philadelphia, 1907 [1].
J. S. Ames, 1897, *Theory of Physics*, Harpers, 1897 [1].

[1] For a systematic presentation of the working principles of mechanics, as developed from the fundamental equation $F/a = F'/a'$, see E. V. Huntington, *The Logical Skeleton of Elementary Dynamics*, American Mathematical Monthly, Vol. 24 (1917), pp. 1–16. Reprints of this article, which a number of teachers have found useful in the class room, may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, O., at ten cents a copy.

J. S. Ames, 1904, *Textbook of General Physics*, Amer. Book Co., 1904 [3].
J. Andrade, *Leçons de mécanique physique*, Paris, 1898 [3].
E. J. Andrews & H. N. Howland, *Elements of Physics*, Macmillan, 1903 [6].
P. Appell, *Mécanique rationnelle*, 3d ed., Vol. 1, Paris, 1909 [3].
F. Auerbach, in A. Winkelmann's *Handbuch der Physik*, 2d ed., I, Art. Grundbegriffe, Leipzig, 1908 [3, 5].
E. Autenrieth, *Technische Mechanik*, Berlin, 1900 [1].
E. H. Barton, *Analytical Mechanics*, Longmans, 1911 [3, 4].
N. H. Black & H. N. Davis, *Practical Physics*, Macmillan, 1913 [2, 7*a*].
L. Boltzmann, *Vorlesungen ueber die Prinzipe der Mechanik*, I, Leipzig, 1897 [3*a*].
E. A. Bowser, *Analytic Mechanics*, 17th ed., Van Nostrand, 1904 [6].
C. A. A. Capito, *Textbook of Mathematics and Mechanics*, Lippincott, 1913 [2].
H. S. Carhart, *Physics for University Students*, I, Allyn & Bacon, 1906 [1].
H. S. Carhart & H. N. Chute, *Physics for High School Students*, 1907 [1].
The Century Dictionary, vol. 6, Article Mass, 1911 [3].
Cheston, Gibson, & Timmermann, *Physics*, Heath, 1906 [1*b*].
I. P. Church, *Statics and Dynamics for Engineering Students*, Wiley, 1886 [1*a*].
O. D. Chwolson, tr. E. Davaux, *Traité de Physique*, I, Paris, 1906 [1, 5].
W. K. Clifford, *Elements of Dynamic*, IV, Macmillan, 1887 [1].
J. H. Cotterill & J. H. Slade, *Applied Mechanics*, Macmillan, 1902 [7*a*].
John Cox, *Mechanics*, Camb. Univ. Press, 1909 [1].
Henry Crew, *General Physics*, Macmillan, 1908 [3].
Henry Crew, *Principles of Mechanics*, Longmans, 1908 [3].
Henry Crew & F. T. Jones, *Elements of Physics*, Macmillan, 1910 [1].
J. A. Culler, *Textbook of Physics*, Lippincott, 1906 [1].
H. M. Dadourian, *Analytical Mechanics*, Van Nostrand, 1913 [6].
A. Daniell, *Textbook of the Principles of Physics*, Macmillan, 1906 [2].
Ch. Delaunay, *Traité de Mécanique rationnelle*, 6th ed., Paris, 1878 [1].
A. P. Deschanel, tr. J. D. Everett, *Natural Philosophy*, I, Appleton, 1893 [1].
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Information in regard to errors of classification or omission will be gratefully received.

BOOK REVIEW.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Projective Geometry. By L. WAYLAND DOWLING. McGraw-Hill Book Company, New York, 1917. 215 pages. $2.00.

If long and expectant waiting for the guest's arrival insures a hearty welcome, this little book first sees the light under most auspicious circumstances. For many years those who have had the good fortune to teach projective geometry have been wishing for a text in English that should lay sufficient emphasis on its non-metrical character and at the same time should be adapted to the powers of the average college junior or even the exceptional younger student. Cremona, excellent book though it is, does not satisfy the first condition; Holgate's translation of Reye has long been out of print; and Veblen and Young's masterful treatise has seemed to many too heavy for the purpose. Here we have a book, small and compact, of pleasing external appearance, well printed in good type, with clear and attractive page broken by figures of reasonable size. Its point of view is non-metrical and yet it does not neglect the metrical applications. It merits careful consideration by all who are interested in this beautiful field of mathematical thought and sympathetic trial in many of the institutions where the subject is taught.

Perhaps the first question confronting the author of an elementary text on projective geometry is whether the conic shall be studied first as generated by projective ranges and pencils or as made up of the self-conjugate elements of a polar field. Shall he ally himself with the school of which we may take Reye as an example or with that typified by Enriques? The logical advantages of the latter have been clearly pointed out,[1] its wisdom from a pedagogical standpoint is not so clear. It takes much longer to reach what to the student seems worth while.

The teacher finds himself, perhaps, in the position of the guide who must choose between two paths up the mountain. There is on the one hand a long and rugged climb, which leads directly to the top, and on which the wonderful landscape bursts suddenly into view in all its beauty; and on the other is an easier path, which half-way up the mountain affords a fine view, if not the equal of that at the higher level. The most hardy of his party would not stop short of the top in any event. Of those who, exhausted, would fail to finish the longer ascent, some, encouraged by the lesser view, will go on to the greater; others, stopping at the half-way house, will at least have something for their labor. Must it not be agreed that unless the party is made up entirely of hardy climbers the wise guide will choose the second route? But is this a true analogy? We confess doubt.

Our author frankly admits in the preface that he chooses Reye for his pattern, and yet, if we may continue the figure, he does not fail to point the way to the higher level. After viewing the beautiful landscape embracing the conic sections and their properties from the level of generation by projective forms, he concludes the book with chapters on projectively related primitive forms of the second kind and polarities in a plane and in a bundle. The question might be raised whether it would have been possible and desirable in one book to give the teacher his choice of paths. Our author has not attempted this. Much of the book would have to be entirely recast if the teacher wished to make his first approach to the conics through the polar field. In particular the involution on a line could not be derived from the involution on a conic.

With what simple means and how quickly are we led to the conics! Only four chapters, 36 pages, are necessary to introduce the student to the fundamental notions, the primitive forms, the theorem of Desargues, the principle of duality, the theorem of perspective quadrangles, the harmonic set, as a preparation for projective one-dimensional primitive forms, the generation of the conic, the theorems of Pascal and Brianchon, and the polar theory of the conic. These are treated in four chapters covering 57 pages. To be sure, the involution is not yet known, nor those properties that depend on it. Three chapters intervene, Chapter IX on the Diameters, Axes, and Algebraic Equations of Curves of the Second Order, Chapter X on Ruled Surfaces of the Second Order, and Chapter XI on Projectively Related Elementary Forms, before in Chapter XII we have the involution defined as a cyclic projectivity of order two. This chapter deals

[1] Charlotte Angas Scott, *Annals of Mathematics*, 2d Series, Vol. 2, pp. 64–72.

with the theory of the involution and imaginary elements, and the following Chapter XIII with the foci and focal properties. This closes the discussion of one-dimensional forms, the remaining fifty pages being occupied with two-dimensional forms, the two chapters mentioned above.

In general outline these first thirteen chapters handle the subject in about the manner that one familiar with Reye would expect. A few features deserve special mention.

In considering the fundamental theorem the author constructs a harmonic scale, calls attention to the fact that harmonic constructions from three points can never yield all the points of a line, though "theoretically we may arrive at a point-row whose points are everywhere dense," and gives a formulation of the Dedekind postulate as here applied. He confesses that the treatment is meager, but it furnishes a starting point for the teacher who considers it advisable to lay stress on the logical foundation of the fundamental theorem.

Defining the involution on a conic as a cyclic projectivity of order two furnishes the author with an occasion for a short discussion of cyclic projectivities in general. The definition of imaginary elements and the solution of certain problems involving their use follow closely the latest edition of Reye's first volume.

Though nearly a quarter of the book is devoted to metrical matters the double ratio receives scant attention.

A more detailed outline of the last two chapters may be in place, since in the selection of material here there is a wider range of possibilities. The definition and determination of perspective and projective transformations of two-dimensional forms are followed by the plane perspectivity, affinity, similitude, and congruence, with a short discussion of the double elements of a collineation. In the last chapter we have the construction and classification of polarities in the plane and bundle; orthogonal and absolute polarity and antipolarity; two polarities in the same plane or bundle with resulting collineation and with application to the cyclic planes and focal axes of cones; quadratic transformations, inversion, circular transformations.

In handling these two chapters the teacher is afforded ample opportunity to leave his impress upon his course. He will wish to amplify the treatment at many points and readjust the emphasis, in order to bring out the advantages of the polar field approach to the conic. One must delay long enough on the summit for the mists to roll away. He will find it necessary even to correct misleading statements such as,[1] "If two planes are collinearly related and have a self-corresponding line, they are in perspective position, or else they are superposed and have in common a sheaf of rays." Evidently what the author means is a line of self-corresponding points rather than a self-corresponding line. Again, in the demonstration of the theorem,[2] "In affinately related planes, the ratio between the areas of corresponding figures is constant," it is implied that corresponding segments bear to each other a ratio that is constant throughout the plane.

[1] Page 165.
[2] Page 170.

Scattered through the book there are some fifty sets of exercises averaging six exercises to the set. For the formulation of these the teacher will be grateful. We have noticed two of them that are faulty in statement:[1] "Show that in the configuration of Desargues *any* line may be taken as an axis of perspectivity. The two triangles and the center of perspectivity will then be uniquely determined," is not clear;[2] "If a hexagon whose vertices are not coplanar nor its three diagonals concurrent is projected from any point on a line which meets all three of the diagonals, show that the lines projecting the vertices are rays of a cone of the second order," draws the conclusion of Pascal from the hypothesis of Brianchon.

The historical notes are few and do not attempt to sketch the development of the subject.[3] "Chasles, *Géométrie Supérieure*, 1880," may be thought misleading, since the date is not that of the first edition.

While the author, as he himself says, has patterned after Reye, he has by no means given us a mere translation. The language and style are the author's, not Reye's. Some things we think might be better said in the interest of clearness and accuracy. For example,[4] "Like primitive forms are each composed of the same kind of elements," appears in its connection to be a definition. The breaks in the argument for the purpose of dualizing are sometimes disturbingly frequent. For some things we should use other names; throw of points or lines where the author uses range and pencil, two-dimensional forms for forms of the second kind, quadratic transformation for quadric transformation. We see no reason for banishing the diagonal point. In passing may we express the hope that some steps will soon be taken to standardize the nomenclature of this subject. It is certainly not an advantage pedagogically to have several different names for the same thing.

It is a pleasure to the teacher to be able to point out to his class the excellences of good figures. This pleasure is in measure denied to the user of the book we are discussing; for while the figures are in general clear and easily read, the teacher, if he calls attention to them at all, will be forced to remark on the carelessness exhibited in their construction. The inconsistency in the use of small circles surrounding designated points and in the use of dotted lines is astonishing. In each of two figures three lines that should pass through a point form a triangle of considerable size. Very unfortunate is Fig. 120, one of the most pretentious in the book. The crude approximations to ellipses here shown are an offence to even the slightly trained eye. What a pity that a book must carry a blemish of this sort, which might so easily have been avoided!

Let us conclude with the hope that this book will find wide acceptance and accomplish much in bringing the subject to the attention of a larger body of students.

JOHN W. BRADSHAW.

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[1] Ex. 5, page 23.
[2] Ex. 9, page 80.
[3] Ex. 6, page 53.
[4] Page 17.

PROBLEMS FOR SOLUTION.[1]

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2660. Proposed by JOSEPH E. ROWE, State College, Pa.

Prove that the distance measured along the side of a triangle, from the point of contact with the inscribed circle to the point of contact with an escribed circle, is equal to the side of the triangle between the two circles.

2661. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find a parallelopipedon whose edges, and the diagonals of its faces, are all rational whole numbers.

2662. Proposed by JOHN LOUKE, New York City.

Assume we have two piles of gold bars. The dimensions of the bars in the first pile are $2.643 \times 5.286 \times 10.573$ and the dimensions of the bars in the second pile are $2.13 \times 6.53 \times 10.573$. If possible, arrange bars from the first pile and from the second pile so as to form perfect cubes, the bars from the piles to be taken separately or in combination.

2663. Proposed by R. P. BAKER, University of Iowa.

From the identity

$$\prod_{k=0}^{\infty}\left(\frac{1}{1-x^{2k+1}}\right) \equiv \prod_{l=1}^{\infty}(1+x^l),$$

Macmahon (*Combinatorial Analysis*, vol. I, p. 10) proves the number of partitions of any integer n into odd parts is equal to the number of partitions of n into parts no two of which are equal.

Taking the classes of some cardinal number, devise an ordinal arrangement which puts each member into one-to-one correspondence with a number of the other class and of such a nature that a direct calculation determines which is the partner of any member in either class.

2664. Proposed by J. W. NICHOLSON, Louisiana State University.

Find the sum of

$$\tfrac{1}{3} - \tfrac{2}{15} + \tfrac{3}{35} - \cdots + (-1)^n \frac{n}{(2n-1)(2n+1)}.$$

2665. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A telegraph wire, which weighs 1/10 of a pound per yard, is stretched between poles on a level ground, so that the greatest dip of the wire is 3 feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

2666. Proposed by W. WOOLSEY JOHNSON, Annapolis, Maryland.

Ten equations between five quantities $x_1, x_2, \cdots, x_5$ being written as follows: $x_1 = 1 - x_2x_4$ and four others formed by cyclic interchange of the suffixes; also $x_5x_1x_2 = x_5 + x_2 - 1$ and four others formed by cyclic interchange; prove that only three of these equations are independent. In other words, the values of x_1 and x_2 being assumed at pleasure, x_3, x_4, and x_5 can be so determined as to satisfy all ten equations.

2667. Proposed by E. L. REES, The University of Kentucky.

Given one diagonal of a parallelogram and the area of the rectangle whose sides are equal to those of the parallelogram, construct the parallelogram so that the diagonal shall make a given angle, α, with a given line and so that the sum of the angles that two adjacent sides make with this line shall be equal to a given angle, β.

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[1] Hereafter the problems will be numbered consecutively beginning at the next number above the total number proposed prior to this date. EDITORS.

2668. Proposed by B. F. FINKEL, Drury College.

Show that

$$v = \frac{2}{9}\frac{ga^2\sigma}{\eta},$$

where a is the radius of a droplet, σ its density, η the viscosity of the air and v the velocity under gravity g. Stokes's Law.

2669. Proposed by S. A. COREY, Albia, Iowa.

Let $A_1, A_2, \cdots, A_8$, and $-(A_1 + A_2 + \cdots + A_8)$ be the vector sides of an enneagon, plane or gauche. Also let $B_1, B_2, \cdots, B_8$, and $-(B_1 + B_2 + \cdots + B_8)$ be the vector sides of a second enneagon, where

$$\begin{aligned}
B_1 &= C_1A_1 - C_2C_5A_3 - C_3C_6A_5 + C_4C_5C_6A_7,\\
B_2 &= C_1A_2 - C_2C_5A_4 - C_3C_6A_6 + C_4C_5C_6A_8,\\
B_3 &= C_2A_1 + C_1A_3 - C_4C_6A_5 - C_5C_6A_7,\\
B_4 &= C_2A_2 + C_1A_4 - C_4C_6A_6 - C_5C_6A_8,\\
B_5 &= C_3A_1 + C_4C_5A_3 + C_1A_5 + C_2C_5A_7,\\
B_6 &= C_3A_2 + C_4C_5A_4 + C_1A_6 + C_2C_5A_8,\\
B_7 &= C_4A_1 - C_5A_3 + C_2A_5 - C_1A_7,\\
B_8 &= C_4A_2 - C_5A_4 + C_2A_6 - C_1A_8,
\end{aligned}$$

C_1, C_2, C_3, C_4, C_5, and C_6 being scalars.

Then, if a_s = tensor A_s, b_s = tensor B_s, and $\cos(A_rA_s)$ = cosine of the angle included between A_r and A_s, and $\cos(B_rB_s)$ = cosine of the angle included between B_r and B_s, establish the following relation between the sides and angles of the two enneagons:

$$\begin{aligned}
&[C_1^2 + C_3C_2^2 + C_6C_3^2 + C_5C_6C_4^2][a_1a_2\cos(A_1A_2) + C_5a_3a_4\cos(A_3A_4)\\
&\qquad + C_6a_5a_6\cos(A_5A_6) + C_5C_6a_7a_8\cos(A_7A_8)]\\
&\qquad = b_1b_2\cos(B_1B_2) + C_5b_3b_4\cos(B_3B_4) + C_6b_5b_6\cos(B_5B_6) + C_5C_6b_7b_8\cos(B_7B_8).
\end{aligned}$$

Show that Geometry problem 506 is a special case of the foregoing. Give illustrative example, using triangle or other simple geometric figure, by assuming that some of the sides of the first enneagon are zero.

SOLUTIONS OF PROBLEMS.

482 (Algebra). Proposed by C. F. GUMMER, Kingston, Ont.

Find a necessary and sufficient condition that the infinite sequences of positive quantities $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ may be such that the series $a_1x_1 + a_2x_2 + \cdots$ and $b_1x_1 + b_2x_2 + \cdots$ either both converge or both diverge, when the x's are any positive quantities.

SOLUTION BY THE PROPOSER.

The condition is that $a_1/b_1, a_2/b_2, \cdots$ all lie between two positive limits m and M $(m < M)$. It is sufficient; for it makes $m < (a_rx_r)/(b_rx_r) < M$. It is necessary; for if it does not hold, either the sequence $(a_1/b_1, a_2/b_2, \cdots)$ or $(b_1/a_1, b_2/a_2, \cdots)$ has $+\infty$ for one of its limits. That is (taking the first case), there is a partial sequence $(a_{i_1}/b_{i_1}, a_{i_2}/b_{i_2}, \cdots)$ of increasing quantities tending to ∞. Hence, using an argument due to DuBois-Reymond, the series

$$\sum_{r=2}^{\infty}\left(\sqrt{\frac{b_{i_{r-1}}}{a_{i_{r-1}}}} - \sqrt{\frac{b_{i_r}}{a_{i_r}}}\right)$$

converges, and the series

$$\sum_{r=2}^{\infty}\frac{a_{i_r}}{b_{i_r}}\left(\sqrt{\frac{b_{i_{r-1}}}{a_{i_{r-1}}}} - \sqrt{\frac{b_{i_r}}{a_{i_r}}}\right)$$

diverges. Hence, if

$$x_{i_r} = \frac{1}{b_{i_r}}\left(\sqrt{\frac{b_{i_{r-1}}}{a_{i_{r-1}}}} - \sqrt{\frac{b_{i_r}}{a_{i_r}}}\right) \qquad (r = 2, 3, \cdots),$$

the partial series $\Sigma_i a_i x_i$ diverges while $\Sigma_i b_i x_i$ converges. Also the other x's may be chosen so small as to make the corresponding values of bx less than the terms of a given convergent series; so that we have made one series diverge while the other converges.

Also solved by S. BEATTY.

483 (Algebra). Proposed by C. R. DUNCAN, Amherst College.

Prove or disprove the following theorem: An infinite series $A_1 + A_2 + A_3 + \cdots + A_n + \cdots$ is convergent or divergent according as

$$\lim_{n=\infty} \frac{A_n}{1 - \frac{A_n}{A_{n-1}}} = 0 \quad \text{or} \quad \neq 0.$$

SOLUTION BY EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

The test when applied to the series

$$1 + \tfrac{1}{1} + 2 + \tfrac{1}{2} + 3 + \tfrac{1}{3} + \cdots + n + \frac{1}{n} + \cdots$$

indicates convergence, whereas the series clearly diverges. If this example be considered artificial consider the series

$$\frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \cdots + \frac{1}{n(\log n)^p} + \cdots.$$

Cauchy's integral test shows that this series diverges when $p \leqq 1$ and converges when $p > 1$.

$$\frac{A_n}{1 - \frac{A_n}{A_{n-1}}} = \frac{1}{\frac{1}{A_n} - \frac{1}{A_{n-1}}} = \frac{1}{n(\log n)^p - (n-1)[\log (n-1)]^p}.$$

Now

$$\frac{1}{(\log n)^p} < \frac{1}{n(\log n)^p - (n-1)[\log (n-1)]^p} < \frac{1}{[\log (n-1)]^p}.$$

Hence, the limit sought by the test is zero for all positive values of p, whereas it should be zero only for values greater than unity.

The suggested test for convergence is therefore no good.

If the limit exists and is greater than zero, then

$$\frac{1}{\frac{1}{A_n} - \frac{1}{A_{n-1}}} > \gamma \quad \text{or} \quad \frac{1}{A_n} - \frac{1}{A_{n-1}} < \frac{1}{\gamma}.$$

Hence

$$\frac{1}{A_{n+p}} - \frac{1}{A_{n-1}} < \frac{p+1}{\gamma},$$

$$A_{n+p} > \frac{1}{\frac{1}{A_{n-1}} + \frac{p+1}{\gamma}} > \frac{\gamma}{\frac{p+1+\gamma}{A_{n-1}}}.$$

(We have assumed a series of positive terms.) The series, therefore, diverges by comparison with the harmonic series. Hence, the test for divergence when applied to series of positive terms is valid.

Also solved by HORACE OLSON.

514 (Geometry). Proposed by VICENTE MILLS, Manila, P. I.

Given an equilateral triangle, the length of the sides being unknown, and a point within, the distances from which to the vertices are given, required the length of a side of the triangle and the angles subtended at the given point by the sides of the triangle.

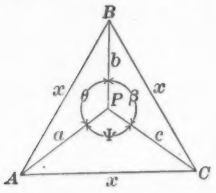
I. Solution by William W. Johnson, Cleveland, Ohio.

From the figure, we have

$$\theta + \beta + \psi = 360°, \qquad \theta = 360° - (\beta + \psi)$$

and

$$\cos\theta = \cos(\beta + \psi) = \cos\beta\cos\psi - \sin\beta\sin\psi, \quad \text{or} \quad \cos\theta - \cos\beta\cos\psi = -\sin\beta\sin\psi.$$



Squaring both sides, we get

$$\cos^2\theta - 2\cos\theta\cos\beta\cos\psi + \cos^2\beta\cos^2\psi = \sin^2\beta\sin^2\psi$$
$$= (1 - \cos^2\beta)(1 - \cos^2\psi)$$
$$= 1 - \cos^2\beta - \cos^2\psi + \cos^2\beta\cos^2\psi.$$

Transposing and uniting, we have

$$\cos^2\theta + \cos^2\beta + \cos^2\psi - 2\cos\theta\cos\beta\cos\psi = 1. \qquad (1)$$

From the cosine formulas in trigonometry, we get

$$\cos\theta = \frac{a^2 + b^2 - x^2}{2ab} \quad (A), \qquad \cos\beta = \frac{b^2 + c^2 - x^2}{2bc} \quad (B), \qquad \cos\psi = \frac{c^2 + a^2 - x^2}{2ca} \quad (C).$$

Substituting these values in (1), clearing of fractions, and arranging terms, we get

$$x^6 - (a^2 + b^2 + c^2)x^4 + (a^4 + b^4 + c^4 - a^2c^2 - b^2a^2 - c^2b^2)x^2 = 0. \qquad (2)$$

Factoring, we find one root is $x^2 = 0$. Removing the factor x^2, we have

$$x^4 - (a^2 + b^2 + c^2)x^2 + a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = 0. \qquad (3)$$

This is a quadratic equation in x^2. Solving equation (3) for x, we find

$$x = \pm\sqrt{\frac{a^2 + b^2 + c^2}{2} \pm \tfrac{1}{2}\sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)}}. \qquad (4)$$

The angles θ, β, and ψ may now be found by inserting the values of x found in (4). The negative signs before the radicals apply when the point P lies outside the triangle.

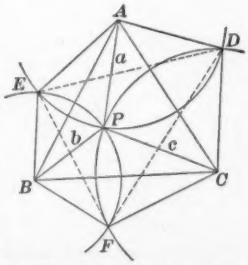
Having found x, the following formulas, adapted for logarithmic computation, may be used to replace (A), (B), and (C).

$$\cos\tfrac{1}{2}\theta = \sqrt{\frac{p(p-x)}{ab}} \quad \text{in which} \quad p = \tfrac{1}{2}(a + b + x),$$

$$\cos\tfrac{1}{2}\beta = \sqrt{\frac{q(q-x)}{bc}} \quad \text{in which} \quad q = \tfrac{1}{2}(b + c + x),$$

$$\cos\tfrac{1}{2}\psi = \sqrt{\frac{r(r-x)}{ac}} \quad \text{in which} \quad r = \tfrac{1}{2}(a + c + x).$$

II. Solution by Daniel Kreth, Wellman, Ia.



Let ABC be the given triangle, and P the given point. Put $AP = a$, $BP = b$, and $CP = c$. Let x = a side of the triangle Then

$$\text{the area} = \frac{x^2}{4}\sqrt{3}. \qquad (1)$$

Construct the triangle AEB = triangle APB, triangle BFC = triangle BPC, and triangle CDA = triangle CPA. Draw DE, EF, and FD.

Since $\angle DAE = 2\angle BAC = 120°$, DE is equal to a side of an equilateral triangle inscribed in a circle whose radius is a.

Hence, $DE = a\sqrt{3}$. In the same manner we find $EF = b\sqrt{3}$ and $FD = c\sqrt{3}$.

$$\text{Area of triangle } DAE = \frac{a^2}{4}\sqrt{3}; \tag{2}$$

$$\text{area of triangle } EBF = \frac{b^2}{4}\sqrt{3}; \tag{3}$$

$$\text{area of triangle } FCD = \frac{c^2}{4}\sqrt{3}; \tag{4}$$

and

$$\text{area of triangle } DEF = \tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}. \tag{5}$$

$$(2) + (3) + (4) + (5) = \text{the area of the polygon } AEBFCD$$

$$= 2 \times \text{area of triangle } ABC.$$

Equating the two expressions for the area of the triangle, we have

$$\frac{x^2}{4}\sqrt{3} = \tfrac{1}{4}(a^2+b^2+c^2)\sqrt{3} + \tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)};$$

$$\therefore\ x = \sqrt{\tfrac{1}{2}\{(a^2+b^2+c^2) + \sqrt{3}[(a+b+c)(b+c-a)(a+c-b)(a+b-c)]\}}.$$

In the triangles we have all the sides given, to find the angles.

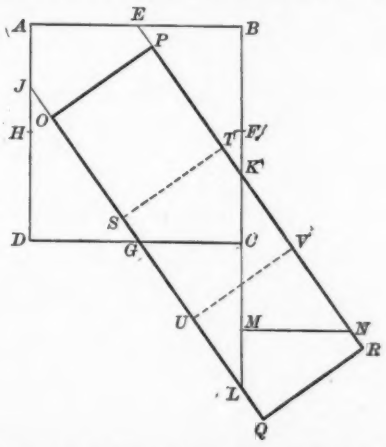
Also solved by NATHAN ALTSHILLER, NELLE L. INGELS and the PROPOSER.

515 (Geometry). Proposed by C. F. GUMMER, Kingston, Ontario.

Show how to cut up a square carpet and make it into three equal square carpets. Estimate the total length of seam in comparison with a side of the original carpet.

I. SOLUTION BY HARRY C. BRADLEY, Massachusetts Institute of Technology.

Construction. Let $ABCD$ be the original square. Take E, F, G, H as the middle points of its four sides. Lay off $BK = DJ$ equal to the distance $EF = GH$. Draw EK and JG, which are obviously parallel. Remove the triangles EBK and JDG, and place them in the positions



GCL and KMN, respectively. In any convenient position draw the line OP perpendicular to JG and EK. Remove the piece $AEPOJ$, and fit it into the position $MNRQL$. The square has now been transformed into the rectangle $OPRQ$, which is three times as long as wide, and may be divided by the lines ST and UV into three equal squares.

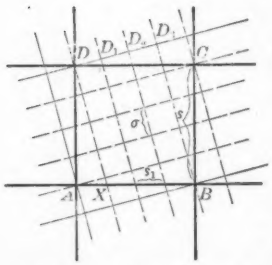
Proof. The area of the rectangle $OPRQ$ is evidently the same as that of the original square, and it is only necessary to prove that the long side, OQ, is three times the short side OP. The side $OQ = JL = JG + GL = JG + EK = 2JG$. To avoid fractions, let the side of the original square equal 6. Then $DG = DH = 3$, $JD = GH$ (by construction) $= 3\sqrt{2}$, and

$$JG = \sqrt{DG^2 + JD^2} = 3\sqrt{3}.$$

Whence, $OQ = 6\sqrt{3}$. The area of the rectangle $OPRQ$ = the area of the square $ABCD = 6 \times 6 = 36$. That is, $OP \times OQ = 36$, or $OP = 2\sqrt{3}$. Hence $OQ = 3OP$.

The Length of Seam. The length of seam, $KL + GC + MN$, is just twice the length of a side of the original square. For $GC + MN$ is the length of a side of the square, since each is equal to half a side, while $KL = KC + CL = KC + BK = \frac{1}{2}BC$.

II. Solution by A. F. Frumveller, Marquette University.



Before attacking this special case, let us deduce the formulas that govern the partition of a square into N squares. Let $ABCD$ be a given square, s its side, and n a number such that $s = ns_1$; in the figure $n = 4$. Draw DX, and complete the figure as indicated. Then we have

(1) $\overline{DX}^2 = s^2 + \frac{s^2}{n^2} = \frac{s^2}{n^2}(n^2 + 1)$,

(2) $\overline{DX} = n\sigma + \frac{\sigma}{n} = \frac{\sigma}{n}(n^2 + 1)$; hence $s^2 = (n^2 + 1)\sigma^2 = N\sigma^2$.

This gives us the relation between the number of sub-squares, and the length of the fundamental segment on AB which produces this partition; $N = n^2 + 1$. The number of complete sub-squares lying within the given square is evidently the greatest integer in

$$\frac{(n-1)(n^2 - n + 1)}{n},$$

which number is the product of

$$\left(\frac{\overline{AB}}{s_1} - 1\right)\left(\frac{\overline{DX}}{\sigma} - 1\right);$$

the shorter formula conveyed by our figure, $(n^2 + 1) - 4n = (n-1)^2$, holds only when n is integral.

The number N is an integer not only when n is integral, but likewise when $n = \sqrt{k}$, in which case $N = k + 1$, and

$$s_1 = \frac{\sqrt{k}}{k}s;$$

if $N = 3$, $n = \sqrt{2}$, and

$$s_1 = \frac{s}{2}\sqrt{2},$$

or half the diagonal of the original square. The length of all the cuts made in the given square, when n is rational, is clearly $2n$ times $\overline{DX}$, or

$$2s\sqrt{n^2 + 1};$$

when n is irrational, it will be noted that the last cut parallel to $\overline{DX}$ within the square (call it $\overline{E_rX_r}$) does not reach the baseline, but crosses the side of the square at a point X_r. By similar triangles, we then have

$$\frac{\overline{DX}}{s_1} = \frac{\overline{E_rX_r}}{s - [ns_1]},$$

where the bracket means as usual, "the greatest integer in ns_1"; hence

$$\overline{E_rX_r} = \overline{DX}\left(\frac{s}{s_1} - \frac{[ns_1]}{s_1}\right) = \overline{DX}(n - [n]),$$

and

$$\Sigma(\overline{D_iX_i}) = \overline{DX}([n] + (n - [n])) = n \cdot \overline{DX}.$$

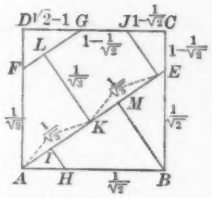
Since the construction of the cross-lines is similar, the sum of all the cuts is as before $2s\sqrt{n^2+1}$.

The skew lattice-work of our figure can be readjusted by slipping downwards each of the columns lying between D and C, until the points $D_1, D_2, \cdots, D_r$ rest on $\overline{DC}$; this is optional for n integral; but for n irrational it is *necessary*, in order to have congruent fragments along the sides of the given square, out of which our N sub-squares may be patched up. Thus we find the actual lines on which the original square must be partitioned. (This problem, for $N = 3$, is solved as above in Sundara Row's "Geom. Exercises in Paper Folding," but without any hint as to how the solution was arrived at.)

The patched squares have "seams" across them, whose total length is $2s$, since two equal segments from the sides combine to form one seam.

III. Solution by the Proposer.

Join AF to BE, GJ to AH, HB to JC and DG, CE to DF.



$$\text{Total seam} = AF + AH + HB + CE$$

$$= \left(\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}\right) AB = 2AB.$$

Also solved by E. B. Escott.

426 (Calculus). Proposed by C. N. Schmall, New York City.

If A be the area of a plane triangle constructed with the sides a, b, c, such that

$$a^3 + b^3 + c^3 = 3k^3,$$

show that the maximum value of A is $\frac{1}{4}k^2$.

Solution by Elijah Swift, University of Vermont.

Using the Heronian formula for the area of a triangle in terms of the sides, we have

$$16A^2 = -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2.$$

Clearly we may make $16A^2$ a maximum instead of A.

By Euler's method of solving an isoperimetric problem, we must equate to zero the partial derivatives with respect to a, b, c, of $16A^2 - 4\lambda(a^3 + b^3 + c^3 - 3k^3)$. (For convenience I use -4λ instead of λ.) In this way we get the equation

$$-4a^3 + 4ab^2 + 4ac^2 - 4\lambda(3a^2) = 0,$$

and two more similar to it. Dividing by $4a$, which cannot be zero,

$$a^2 + 3\lambda a = b^2 + c^2 \tag{1}$$

and two equations, (2) and (3), obtained by permuting cyclically the letters a, b, c in (1).

But from (1) a must satisfy the equation

$$2x^2 + 3\lambda x = a^2 + b^2 + c^2, \tag{4}$$

and (2) and (3) show that b and c are roots of the same equation. Since this equation has only one positive root, a, b, c must all be equal, and the triangle equilateral, whence it appears that its area must be $\frac{1}{4}k^2\sqrt{3}$, which is not the value given in the statement.

The same method used here shows that this result holds when the auxiliary condition is $a^n + b^n + c^n = 3k^n$, $n > 1$.

It is evident from the conditions of the problem that the values found give a maximum.

Also solved by R. A. Johnson, S. A. Corey, and J. B. Reynolds.

427 (Calculus). Proposed by ROGER S. JOHNSON, Adelbert College, Cleveland, Ohio.

Of all ellipses circumscribed about a given parallelogram, the minimum [maximum], with regard to area, has as conjugate diameters the diagonals of the parallelogram.

SOLUTION BY HENRY D. THOMPSON, Princeton, N. J.

Let the given parallelogram be $PQP'Q'$, with $QP = 2h$, and $Q'P = 2k$, let the diagonals $P'P$, $Q'Q$ intersect at O, and through O take the x-axis parallel to, and positive in, the direction of QP and $P'Q'$, and the oblique y-axis parallel to, and positive in, the direction $Q'P$ and $P'Q$, and let the angle of the axes be ω. Take a representative ellipse through $PQP'Q'$ cutting the positive x-axis in the point U, with the coördinates (u, o), and the y-axis in the point $V(o, v)$. Then the equation of this ellipse is

$$\frac{x^2}{u^2} + \frac{y^2}{v^2} = 1, \tag{1}$$

and since it passes through $P(h, k)$ the relation holds

$$\frac{h^2}{u^2} + \frac{k^2}{v^2} = 1. \tag{2}$$

The area for the ellipse is

$$uv\pi \sin \omega, \tag{3}$$

and this is a minimum when

$$M = uv \tag{4}$$

is a minimum. Differentiation of (4) and (2) gives for the minimum $dM/du = v + u dv/du = 0$, with $-h^2/u^3 - (k^2/v^3)dv/du = 0$, and elimination of dv/du gives $k^2/v^2 = h^2/u^2 = \frac{1}{2}$ from (2), whence $u = h\sqrt{2}$, $v = k\sqrt{2}$. These values set in d^2M/du^2 give $4k/h$, a positive magnitude, and the area of the ellipse is a minimum. This ellipse is $x^2/2h^2 + y^2/2k^2 = 1$, and the tangents at the vertices evidently intersect on the diagonals $P'P$ and $Q'Q$, which are thus conjugate, since OU and OV are conjugate.

The theorem can be proved without the use of the calculus as follows: The coördinates of a point on (1) can be expressed by means of a parameter φ by the equations $x = u \cos \varphi$, $y = v \sin \varphi$ which for the point $P(h, k)$ become $h = u \cos \varphi_1$, $k = v \sin \varphi_1$, where φ_1 changes with u and v. Then (4) becomes $M = hk/\sin \varphi_1 \cos \varphi_1 = 2hk/\sin 2\varphi_1$, which has the single minimum when $\sin 2\varphi_1$ is a maximum, that is unity, or when $\varphi_1 = \pi/4$, or when $u = h\sqrt{2}$, $v = k\sqrt{2}$, as before.

Also solved by MARIE WHELAN, J. B. REYNOLDS, and FLORENCE P. LEWIS.

Several solutions of Mechanics problem 339 will be printed in an early issue. EDITORS.

340 (Mechanics). Proposed by PAUL CAPRON, U. S. Naval Academy.

A rigid straight line l passes through a fixed point O, but is otherwise free to move in a plane. If C is the instantaneous center of rotation for l, prove that CO is always perpendicular to l and that, if (O being used as pole) $\rho = f(\theta)$ represents the locus of any point P on l, OC is always equal to $(d/d\theta)f(\theta)$.

SOLUTION BY S. W. REAVES, University of Oklahoma.

If we know the directions of motion at a given instant of two points of the moving line, then the center of rotation for that instant is to be found at the intersection of the normals drawn at these points to their directions of motion. (See Ziwet, *Theoretical Mechanics*, Art. 23; Demartres, *Cours de Géométrie infinitésimale*, Art. 20.)

In the problem before us the direction of motion of the point of l, which at the given instant coincides with O, is along l. Hence the normal to l at O must pass through C.

Again, the normal at P to the path of P, $\rho = f(\theta)$, must also pass through C. Hence the angle OPC is the complement of the angle ψ between OP and the tangent at P. Hence,

$$\tan OPC = \cot \psi = \frac{d\rho}{\rho d\theta}.$$

(See any book on Calculus.) Therefore, from the right triangle COP, we have

$$OC = \rho \tan OPC = \frac{d\rho}{d\theta} = \frac{d}{d\theta} f(\theta).$$

Also solved by J. B. REYNOLDS and W. E. CEDERBERG.

257 (Number Theory). Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Find a general expression for the number of positive integers from 1 to 10^t, inclusive, every one of which contains the figure 9 exactly r times $(0 \leqq r \leqq t)$.

SOLUTION BY THE PROPOSER.

In the case of the integers from 1 to 10, we have nine which do not contain the figure 9 and one which contains one 9. This shall be indicated by the expression $9 + 1$.

In the case of 10^2, the number of integers, which do not contain 9, is $9 \cdot 9$, or 9^2; which contain one 9, is $9 \cdot 1 + 9$, or $2 \cdot 9$; which contain two 9's, is 1, and we have the expansion of

$$(9 + 1)^2 = 9^2 + 2 \cdot 9 + 1.$$

For 10^3, we have $9 \cdot 9^2$, $9 \cdot 2 \cdot 9 + 9^2$, $9 \cdot 1 + 2 \cdot 9$, and 1, or $9^3 + 3 \cdot 9^2 + 3 \cdot 9 + 1$.

Then, for 10^k, assume the expansion of $(9 + 1)^k$, or

$$9^k + \binom{k}{1} 9^{k-1} + \binom{k}{2} 9^{k-2} + \cdots + \binom{k}{n-1} 9^{k-(n-1)} + \binom{k}{n} 9^{k-n} + \cdots + \binom{k}{k-1} 9 + 1.$$

For 10^{k+1} we reason as follows: The number of integers which do not contain 9 is $9 \cdot 9^k$, or 9^{k+1}; which contain one 9, is $9 \cdot \binom{k}{1} 9^{k-1} + 9^k$, or $\binom{k+1}{1} 9^k$; which contain two 9's, is $9 \cdot \binom{k}{2} 9^{k-2} + \binom{k}{1} 9^{k-1}$ or $\binom{k+1}{2} 9^{k-1}$, and which contain n 9's, is

$$9 \cdot \binom{k}{n} 9^{k-n} + \binom{k}{n-1} 9^{k-(n-1)} = \left[\binom{k}{n} + \binom{k}{n-1}\right] 9^{k-n+1} = \binom{k+1}{n} 9^{k+1-n}.$$

Hence, we have, for 10^{k+1}, the expansion of $(9 + 1)^{k+1}$, or

$$9^{k+1} + \binom{k+1}{1} 9^k + \cdots + \binom{k+1}{n} 9^{k+1-n} + \cdots + \binom{k+1}{k} 9 + 1.$$

Now, the derived expression holds for $k = 2$ and for $k = 3$; hence it holds for all positive integral values of k.

Therefore, the general expression required is $\binom{t}{r} 9^{t-r}$.

Also solved by HORACE OLSON, H. C. FEEMSTER, C. C. YEN, and N. P. PANDYA.

258 (Number Theory). Proposed by A. A. BENNETT, University of Texas.

Find a recursion formula in terms of binomial coefficients for a_n, where the a's are defined by the condition that the persymmetric determinants

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{n-1} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot \\ a_2 & a_3 & \cdot & \cdot & \cdot \\ a_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix}$$

are each equal to unity for every positive integer n.

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Though this solution does not directly involve binomial coefficients, yet by finding the value of a_n it may be considered to dispose of the problem sufficiently.

The given conditions show that $a_0 = a_1 = 1$, and that the other a's may be found in succession uniquely from equations in which they appear with the coefficient unity. The a's being determinate there exists a sequence $x_0, x_1, \cdots$ such that

$$a_n = a_{n-1}x_0 + a_{n-2}x_1 + \cdots + a_0x_{n-1}, \qquad n = 1, \ , \cdots; \tag{1}$$

for the first n equations of (1) have a determinant equal to unity.

If we apply the substitutions (1) to the last row of

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n+1} & \cdot & \cdot & a_{2n} \end{vmatrix},$$

and simplify by means of the other rows, the last row becomes

$$0,\ a_0x_n,\ a_0x_{n+1} + a_1x_n,\ a_0x_{n+2} + a_1x_{n+1} + a_2x_n,\ \cdots.$$

With similar treatment, the preceding row becomes

$$0,\ a_0x_{n-1},\ a_0x_n + a_1x_{n-1},\ \cdots,$$

and so for all but the first row. On simplifying by columns we get, since $a_0 = 1$,

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdot & \cdot \\ x_2 & x_3 & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{2n-1} \end{vmatrix} = 1.$$

A like treatment of the other determinant gives

$$\begin{vmatrix} x_0 & x_1 & x_2 & \cdot & \cdot \\ x_1 & x_2 & \cdot & \cdot & \cdot \\ x_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{2n} \end{vmatrix} = 1.$$

Hence, $x_2, x_3, \cdots$ are defined in terms of x_0, x_1 in the same way as $a_2, a_3, \cdots$ in terms of a_0, a_1.

Also $a_0 = a_1 = 1$, $a_2 = 2$.

Hence, $x_0 = x_1 = 1$, $x_2 = 2$, by direct calculation.

Hence, $x_n = a_n$. Hence, (1) becomes

$$a_n = a_{n-1}a_0 + a_{n-2}a_1 + \cdots + a_0a_{n-1}, \qquad n = 1, 2, \cdots. \tag{2}$$

To calculate a_n, we infer from (2) that the coefficient of t^n in $u \equiv a_0 + a_1t + a_2t^2 + \cdots$ is equal to the coefficient of t^{n-1} in u^2, when $n = 1, 2, \cdots$.

Hence, $(u - 1)/t = u^2$.

Hence, $u = 1/2t - \sqrt{1 - 4t}/(2t)$, the minus sign being necessary to make u a series in positive powers of t.

Hence, the coefficient a_n of t^n in u equals $\dfrac{\lfloor 2n}{\lfloor n \ \lfloor n+1}$.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO A SOLUTION OF THE BIQUADRATIC EQUATION.

BY ENRIQUE CRUCHAGA, Chilean Consul, Philadelphia, Pennsylvania.

Soit $x^4 + px^3 + qx^2 + rx + s = 0$ l'équation à résoudre et x_0, x_1, x_2, x_3 les racines à chercher. Cherchons u, v, y, z, racines des fonctions symétriques suivantes:

$$z = x_0x_1, \qquad y = x_2x_3, \qquad u = x_0 + x_1, \qquad v = x_2 + x_3.$$

On peut aisément établier:

$$\text{I. } q - (z + y) = uv, \qquad \text{II. } yz = s,$$
$$-p = u + v. \qquad uy + vz = -r.$$

On déduit du système I:

$$u = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q + z + y}; \qquad v = -\frac{p}{2} \mp \sqrt{\frac{p^2}{4} - q + z + y},$$

et, en substituant ces valeurs dans le système II, et en simplifiant,

$$y - z = \frac{-r + \frac{p}{2}(y + z)}{\sqrt{\frac{p^2}{4} - q + y + z}};$$

en élevant cette équation au carré et en l'ajustant $4yz = 4s$, on obtient la résolvant aisément,

$$(y + z)^3 - q(y + z)^2 + (pr - 4s)(y + z) - [s(p^2 - 4q) + r^2] = 0.$$

En combinant les trois valeurs de $(y + z)$ avec celle de $yz = s$, nous obtiendrons six valeurs des produits $x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3$. En substituant dans le système I les trois valeurs des $(y + z)$, on obtiendra les six valeurs des sommes, $x_0 + x_1, x_0 + x_2, \cdots$, etc., lesquelles en combinaison avec celles des produits $x_0x_1, x_0x_2, \cdots$, etc., nous donneront définitivement les quatre valeurs cherchées.

On peut connaître les six valeurs possibles des produits $uyvz = (q - z - y)s$, $uy + vz = -r$. On fera ensuite la combinaison alternée des facteurs de deux valeurs conjuguées de u, y, v, z.

II. Relating to the Definition of a Function E.

By Oscar Schmiedel, Bellevue College, Bellevue, Nebraska.

A function E is defined by the following three statements:

1. $E^{(l,n,x)}_{[r]} = 1 + \frac{x}{l} + \frac{x^2}{l(l+n)} + \cdots + \frac{x^r}{l(l+n)\cdots(l+\overline{r-1}n)}.$

2. $E^{[r]}_{(l,n,x)} = -\frac{l-n}{x} - \frac{(l-n)(l-2n)}{x^2} - \cdots - \frac{(l-n)(l-2n)\cdots(l-\overline{r-1}n)}{x^{r-1}}.$

3. $E^{(x^n)} = 1 + \frac{nx^n}{1} + \frac{(nx^n)^2}{1+n} + \frac{(nx^n)^3}{(1+n)(1+2n)} + \cdots.$

The second statement is a consequence from the first when r is negative; the third a special definition for the purpose of simplification in the present remarks. The index r in the first and second may be omitted when for particular values of l and n the series either terminates or becomes infinite; the index l may also be omitted when $l = 1$; thus, $E^{(1,x)} = e^x$.

It will now appear by reference to the September, 1917, number of the Monthly, pp. 342–343, that[1] $\int \sin\theta^2 d\theta$ and $\int \cos\theta^2 d\theta$ may be written in the following concise forms:

$$\int \sin\theta^2 d\theta = -\theta \sum_{t=0}^{\infty} \prod_{s=1}^{s=t} \frac{2\theta^2}{1+2s} \cdot \sin\left(t\frac{\pi}{2} - \theta^2\right)$$

$$= \theta \sin\theta^2 \left(1 - \frac{(2\theta^2)^2}{3\cdot 5} + \frac{(2\theta^2)^4}{3\cdot 5\cdot 7\cdot 9} - + \cdots\right) - \theta\cos\theta^2\left(\frac{2\theta^2}{3} - \frac{(2\theta^2)^3}{3\cdot 5\cdot 7} + \frac{(2\theta^2)^5}{3\cdot 5\cdots 11} - + \cdots\right)$$

$$= \tfrac{1}{4}i\theta^{-1}\sin\theta^2(E^{(-i\theta^2)} - E^{(i\theta^2)}) + \tfrac{1}{4}\theta^{-1}\cos\theta^2(E^{(-i\theta^2)} + E^{(i\theta^2)}) - \tfrac{1}{2}\theta^{-1}\cos\theta^2$$

$$= \tfrac{1}{4}\theta^{-1}(\cos\theta^2 + i\sin\theta^2)E^{(-i\theta^2)} + \tfrac{1}{4}\theta^{-1}(\cos\theta^2 - i\sin\theta^2)E^{(i\theta^2)} - \tfrac{1}{2}\theta^{-1}\cos\theta^2$$

$$= \tfrac{1}{4}\theta^{-1}(e^{i\theta^2}E^{(-i\theta^2)} + e^{-i\theta^2}E^{(i\theta^2)}) - \tfrac{1}{2}\theta^{-1}\cos\theta^2;$$

$$\int \cos\theta^2 d\theta = \tfrac{1}{4}i\theta^{-1}(e^{i\theta^2}E^{(-i\theta^2)} - e^{-i\theta^2}E^{(i\theta^2)}) + \tfrac{1}{2}\theta^{-1}\sin\theta^2.$$

If in a similar manner the integral of $e^{\theta^2}d\theta$ is sought by means of the reduction

[1] We write $\sin\theta^2$ for $\sin(\theta^2)$, etc.

formula

$$\int \theta^m e^{\theta^n} d\theta = \frac{1}{n}\,\theta^{m-n+1} e^{\theta^n} - \frac{m-n+1}{n} \int \theta^{m-n} e^{\theta^n} d\theta,$$

the following two forms are obtained:

$$\int \theta^m e^{\theta^n} d\theta = \frac{1}{n}\,\theta^{m-n+1} e^{\theta^n} \sum_{t=0}^{t=r} \prod_{s=1}^{s=t} \frac{sn-m-1}{n\theta^n} + \prod_{s=1}^{s=r+1} \frac{sn-m-1}{n} \cdot \int \theta^{m-(r+1)n} e^{\theta^n} d\theta,$$

$$\int \theta^m e^{\theta^n} d\theta = -\frac{1}{n}\,\theta^{m-n+1} e^{\theta^n} \sum_{t=1-r}^{t=-1} \prod_{s=t+1}^{s=0} \frac{n\theta^n}{sn-m-1} + \prod_{s=2-r}^{s=0} \frac{n}{sn-m-1} \cdot \int \theta^{m+(r-1)n} e^{\theta^n} d\theta,$$

where r is a positive integer.

The second of these gives, for $m = 0$, $n = 2$, the desired development as a convergent series, which by 3. takes the form

$$\int e^{\theta^2} d\theta = -\tfrac{1}{2}\theta^{-1} e^{\theta^2} \left(-\frac{2\theta^2}{1} + \frac{(2\theta^2)^2}{3} - \frac{(2\theta^2)^3}{3\cdot 5} + - \cdots \right)$$

$$= -\tfrac{1}{2}\theta^{-1} e^{\theta^2} E^{(-\theta^2)} + \tfrac{1}{2}\theta^{-1} e^{\theta^2}.$$

Comparison of these three results shows the following relation to hold:

$$\int \cos\theta^2 d\theta + i \int \sin\theta^2 d\theta = \int e^{i\theta^2} d\theta,$$

as was to be expected.

When $m+1$ is a multiple of n, the development of $\int \theta^m e^{\theta^n} d\theta$ is a finite series. Thus, let $m = 7,\ 0$; $n = 2,\ \frac{1}{4}$, respectively, and $r = 3$, and the first formula gives:

$$\int \theta^7 e^{\theta^2} d\theta = \tfrac{1}{2}\theta^6 e^{\theta^2} \sum_{t=0}^{t=3} \prod_{s=1}^{s=t} (s-4)\theta^{-2}$$

$$= -3e^{\theta^2} \left(1 - \frac{\theta^2}{1!} + \frac{\theta^4}{2!} - \frac{\theta^6}{3!} \right)$$

$$= \tfrac{1}{8}\theta^8 e^{\theta^2} E_{(1/5,\,-1/5\theta^2)} \qquad \text{[by 2];}$$

$$\int e^{\theta^{1/4}} d\theta = 4\theta^{3/4} e^{\theta^{1/4}} \sum_{t=0}^{t=3} \prod_{s=1}^{s=t} (s-4)\theta^{-1/4}$$

$$= -4!\, e^{\theta^{1/4}} \left(1 - \frac{\theta^{1/4}}{1!} + \frac{\theta^{2/4}}{2!} - \frac{\theta^{3/4}}{3!} \right)$$

$$= \theta e^{\theta^{1/4}} E_{(1/5,\,-1/5\theta^{1/4})} \qquad \text{[by 2].}$$

It is interesting to observe that these values of m and n substituted in the second of the formulae above will give, after a little transformation, the same finite

forms; thus, in the latter case:

$$\int e^{\theta^{1/4}} d\theta = -4! e^{\theta^{1/4}} \left(1 - \frac{\theta^{1/4}}{1} + \frac{\theta^{2/4}}{2!} - \frac{\theta^{3/4}}{3!}\right) + 4!,$$

differing from the previous result only by a constant term.

This result is general, and gives, when the exponent of the argument is a unit fraction:

$$\int e^{\theta^{1/n}} d\theta = (-1)^{n-1} n! e^{\theta^{1/n}} \sum_{t=0}^{t=n-1} (-1)^t \frac{\theta^{t/n}}{t!},$$

or, the same expression plus the term $(-1)^n n!$.

III. Relating to Magic Squares for the New Year, 1918.

By S. A. Corey, Albia, Iowa.

The two following 4×4 magic squares are taken indiscriminately one from each of two large families of 4×4 magic squares with sum 1918 which may be formed in like manner from the two sets of 15 square numbers here employed. It is improbable that other families of 4×4 magic squares with the sum 1918 can be found with similar characteristics.

464	383	513	558
591	480	482	365
414	429	527	548
449	626	396	447

(*A*)

461	409	516	532
565	483	508	362
440	426	501	551
452	600	393	473

(*B*)

Some of the peculiarities of the *families* of magic squares from which (*A*) and (*B*) are taken are as follows:

Each number employed may be decomposed into the sum of the squares of *four* separate numbers taken from the series of natural numbers, 1 to 18 inclusive (18 being the last two figures of 19*18*).

In forming such sums *all* the square numbers used are used just *four* times with *one* exception in each case as follows:

In (*A*) 10 is used *eight* times, and
In (*B*) 6 is used *eight* times.

In forming the numbers employed in (*A*) all the natural numbers, 1 to 18 inclusive, are used, except 1, 11 and 13, the sum of which is 25.

In forming the numbers employed in (*B*) all the natural numbers, 1 to 18 inclusive, are used, except 5, 9 and 11, the sum of which is likewise 25. To illustrate:

$2^2 + 6^2 + 10^2 + 18^2 = 464$, $3^2 + 7^2 + 10^2 + 15^2 = 383$, $4^2 + 8^2 + 12^2 + 17^2 = 513$, $5^2 + 9^2 + 14^2 + 16^2 = 558$ [464, 383, 513 and 558 being the numbers in

the first row of (A)]; $1^2 + 6^2 + 10^2 + 18^2 = 461$, $2^2 + 6^2 + 12^2 + 15^2 = 409$, $3^2 + 7^2 + 13^2 + 17^2 = 516$, $4^2 + 8^2 + 14^2 + 16^2 = 532$ [461, 409, 516 and 532 being the numbers in the first row of (B)].

We note above that the sum of the natural numbers not used in (A) is 25 and that 10 is used more than the usual number of times. Similarly in (B) the numbers 25 and 6 are exceptional numbers. Were we inclined to be superstitious we might therefore say that October 25 (10–25) and June 25 (6–25) are dates of transcendent importance in 1918, say, the date of the signing of the treaty of peace, and the date of cessation of fighting in the world war.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

It is designed that this new department of the MONTHLY shall record the details of organization and activity of each undergraduate mathematics club in America, shall serve as a medium of communication between such clubs, and shall contain information helpful and suggestive for their guidance, and for the development of their usefulness.

As soon as the information can be collected a list of the clubs will be published. To 35 clubs already discovered appeal has been made for details, a portion of which will be given in connection with this list. It is earnestly desired that every other club shall make itself known to the editor, reporting on such things as: (1) the club's exact name, (2) the date (year and month) of organization, (3) the club's object and those eligible for membership, (4) the names of the officers of the club for 1917–18, (5) the number of members of the club, the average attendance, and the number of meetings held each year, (6) the dates of meetings in 1917–18, the titles of papers read and the names of the speakers. The editor would like to receive also copies of all printed programs for the current and earlier years.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, 1917–18.

Membership is open: (1) to those who have had or are taking a course in analytic geometry; (2) to the members of the freshman class who win the prizes awarded in connection with the competitive examination on original problems in entrance mathematics; and (3) to mathematical students in the second semester of the freshman year who have attained to the higher honor grade in a mathematical course of the first semester. Average attendance: 52.

Chairman: Professor N. F. Davis.

Committee on Program: Professor R. G. D. Richardson, Marion R. Luther Gr., Mary I. Briggs '18, Clarence R. Adams '18, James B. Hobbs '18.

Committee on Arrangements: Doctor T. H. Brown, Lydia L. Cooper Gr., Esther E. Brintzenhoff '19, Albert S. Pratt '18, Chauncey D. Wentworth '20.

The program for the year 1917–18 is as follows:

November 16: "Geometric Exercises in Paper Folding" by May Sperry '18; "Mathematical Fallacies" by Albert S. Pratt '18;

December 14: "Growth and Form" by Mary I. Briggs '18; "Secrets of Lightning Calculators" by James B. Hobbs '18;

January 11: "Nomography" by Professor E. V. Huntington of Harvard University;

February 15: "Mathematics of Warfare" by Mary E. Carroll '19; "Probability Curves" by Clarence R. Adams '18; "The Development of Mathematical Symbolism" by Marion R. Luther Gr.;

March 15: "The Russian Peasant Method of Multiplication and the Binary Scale of Notation" by Elsie M. Flint '18; "Non-Euclidean Geometry" by Professor H. P. Manning;

April 19: "Some Philosophical and Psychological Implications of the Game of Nim" by E. B. Delabarre, Professor of Psychology in Brown University;

May: Picnic.

THE JUNIOR MATHEMATICAL CLUB OF THE UNIVERSITY OF CHICAGO.

Professor Slaught has kindly furnished most of the following notes:

"This club was organized by students in the autumn of 1905 chiefly through the instrumentality of N. J. Lennes, who was then a graduate student at the university. Its purpose was to afford an opportunity to all students primarily interested in mathematics, whether undergraduates or graduates, to present before a sympathetic audience the results of reading and investigation, especially along lines not likely to be included in regular class work. The intention, for the most part well carried out, has been, and is, to keep these papers strictly elementary as compared with the papers presented in the Graduate Mathematical Club of the university where only the results of original research are given. In fact, the chief incentive to organizing this Junior Club was the great discouragement experienced by even the younger graduate students (not to mention the undergraduates) in trying to be interested and to look wise while attending the meetings of the Graduate Club.

"The meetings of the Junior Club occur every second week during the year from October to June, alternating with those of the Graduate Club. They rarely last for more than an hour and a quarter, the half hour preceding the presentation and discussion of papers being usually devoted to friendly intercourse encouraged by a cup of tea.

"The chief results of these meetings, looked at in perspective over a decade, are: (1) Accumulation of information by the members in many lines that they might otherwise not find time or inclination to look up for themselves; (2) stimulation to activity on the part of individuals who might otherwise be content to do the required class-room work and nothing more; (3) cultivation of independence in study, and ease and clearness in presenting the results of study to an audience; (4) preparation of students either for teaching or for further advanced study,

in many ways which would not otherwise be realized in the regular college and university work.

"The average number of members is somewhere between fifteen and twenty. There is no constitution and no requirement for membership other than a devotion to mathematics and a desire to commune with others having the same devotion. The minutes of the first meeting record that 'Professor Slaught was made an ex officio member with the power of veto' to make sure, I suppose, that things would go straight. I have almost always given the opening address at the autumn meeting.

"It may be of interest to add that the first president of the Junior Club was W. R. Longley, now professor at Yale, and that the first program committee included G. D. Birkhoff, now professor at Harvard, and Dr. H. F. McNeish, now of the De Witt Clinton High School, New York."

Officers, 1917–18: President, E. P. Lane; vice-president, C. Gouwens; secretary, Minna Schick.

The program committee consists of the officers and of I. A. Barnett, W. G. Simon and Gladys Gibbons.

All officers and members of the committee are graduate students.

The following is the program for the fall quarter, 1917:

October 12: Election of officers. "Purposes and Aims of the Club" by Professor Slaught.

October 24: "G. Cantor" by Minna Schick; "Notion of Number" by I. A. Barnett;

November 7: "Unique Factorization in the Quadratic Realm" by C. Gouwens;

November 21: "Finite Projective Geometries" by Cyril A. Nelson, Jr.;

December 5: "History of the Teaching of Collegiate Mathematics in the United States" by W. G. Simon.

The Mathematics Club of the University of Kansas, 1917–18.

Membership is open only to those who have had a three-hour year course in calculus. Average attendance about 19.

Officers: President, Mildred Abel '18; vice-president, Eran Burgert '18; secretary, Frances Adams '18; faculty adviser, Professor U. G. Mitchell.

Program Committee: Helen Garman '18, Mary Smith '18, and Edward Buffington '18.

The program for 1917–18 is as follows:

October 8: "Air Planes" by Professor Lefschetz;

October 22: "Poincaré's Non-euclidean World" by Mildred Abel '18;

October 31: "Fortunes told mathematically" by Professor Van der Vries;

November 12: "The Number Zero" by Helen Wedd '18; "The Number π" by Beulah Armstrong Gr.;

November 26: "Feuerbach's Circle" by Mary Smith '18; "A Probability Machine" by Anna Marm Gr.;

December 10: "Mathematics of the War" by Eran Burgert '18;

January 7: "Numerical Properties of Color and Sound" by Edward Buffington '18;
January 21: "Mathematical Fallacies" by Sarah Bingham '18; "Card Tricks" by Edith Whitcher '19;
February 11: "Mathematics as a Fine Art" by Georgia Beebe '18; "Mathematical Accomplishments of Women" by Irma Leon '19;
February 25: "History of Time Pieces" by Wealthea Babcock '19; "History of the Metric System" by Sadie Horsley Gr.;
March 11: "Zeno's Paradoxes" by Frances Adams '18;
March 25: "Addition and Subtraction by Logarithms" by E. B. Miller (Instructor);
April 8: "Mathematicians who became Famous in other Fields" by J. H. Hoover Gr.;
April 22: "Regular Polygons inscribable in a Circle" by Goldie Piper '19;
May 12: "The Beginnings of Higher Mathematics in England" by Faye Doddridge '19; "The Beginnings of Mathematics in the United States" by Helen Garman '18;
May 26: Annual Picnic.

TOPICS FOR CLUB PROGRAMS.

1. The Oldest Mathematical Work Extant.

This is the heiratic papyrus said to have been one of a number found at Thebes in the ruins of a small building near the Ramesseum. It was purchased at Luxor in 1858 by A. Henry Rhind and after his death it passed into the hands of a gentleman from whom it was purchased by the trustees of the British Museum in 1864.

The papyrus was copied by a scribe named Ahmes, between 2000 B. C. and 1700 B. C., from an older work. The text comprises a series of propositions or problems in arithmetic, mensuration, trigonometry, and in the various branches of practical geometry, sometimes accompanied by diagrams, representing the class of practical mathematical knowledge which an overseer of royal farms, or a revenue officer or the master-mason employed in building a pyramid or temple, would be expected to possess.

Any approximately complete bibliography of the papyrus should contain more than forty titles. Three may be regarded as referring to works which are fundamental in connection with its study. These are: (1) The facsimile in original colors published by the British Museum in 1898 (21 plates, 15 x 25 in.) with a preface from which most of the details mentioned above have been taken; (2) the translation into German, and commentary, prepared by the Heidelberg aegyptologist August Eisenlohr with the aid of two mathematicians, his brother Friedrich and Moritz Cantor; the second edition, without plates (a work in quarto format of nearly 300 pages), published by Hinrich's Buchhandlung in Leipzig under the title: *Ein mathematisches Handbuch der alten Aegypter* (*Papyrus Rhind des British Museum*); (3) the discussion of Eisenlohr's

work by L. RODET, "Les prétendus problèmes d'algèbre du Manuel du Calculateur égyptien," in *Journal asiatique*, Paris, 1881–82, série 7, tome 18; these articles were reprinted in 1882 (122 pages).

Three references may be given to historical works: (1) M. CANTOR, *Vorlesungen über Geschichte der Mathematik*, Band I, 3. Auflage, Leipzig, Teubner, 1907, pages 57 ff.; (2) S. GÜNTHER, *Geschichte der Mathematik*, I. Teil, Leipzig, Göschen, 1908, pages 24–35; (3) W. T. SEDGWICK and H. W. TYLER, *A Short History of Science*, New York, Macmillan, 1917, pp. 30–34.

In English there are also the papers by (1) F. L. GRIFFITH, in *Proceedings of the Society of Biblical Archaeology*, London, volumes 13 and 16, 1891, and 1894; and (2) G. A. MILLER, in *School Science and Mathematics*, Chicago, volume 5, 1905.

In French: (1) V. V. Bobynin, (a) "Sur le procédé employé dans le papyrus Rhind pour réduire les fractions en quantièmes," *Bibliotheca Mathematica*, Leipzig, 2. Reihe, Band 4, 1890; (b) "Méthode expérimentale dans la science des nombres et principaux résultats obtenus," *L'Enseignement Mathématique*, Paris-Genève, tome 8, 1906; (2) G. MILHAUD, *Nouvelles études sur l'histoire de la pensée scientifique*, Paris, Alcan, 1911, pp. 58–66; (3) L. BRUNSCHVICG, *Les étapes de la philosophie mathématique*, Paris, Alcan, 1912, pp. 26–32.

2. GEOMETROGRAPHY AND OTHER METHODS OF MEASUREMENT OF GEOMETRICAL CONSTRUCTIONS.

As far back as 1833 Steiner wrote a passage, often quoted, regarding the desirability of an investigation as to the simplicity and exactitude of geometric constructions;[1] but it is only within the past thirty years that theories along these lines have been developed.

Geometrography may be defined, in the words of its inventor Émile Lemoine (1840–1912), as "the art of geometrical constructions." Its aim is to discover which of the various ways of solving a problem is the simplest, or, in other words, which way requires us to perform the smallest number of operations.

Lemoine's first memoir on the subject was read before the French Association for the Advancement of Science in 1888 and during the next twenty years he published more than thirty papers and notes on the subject. The theory is pretty well summed up in his little book: *Géométrographie ou art des constructions géométriques* (Scientia no. 18). Paris, Gauthier-Villars, 1902.

Reference may also be given to: (1) Lemoine's papers on a geometrographic comparison of twelve constructions deduced from eleven solutions of the same problem in *Comptes rendus de l'Association Française pour l'Avancement des Sciences*, 1899 and 1900; to (2) Lemoine's revision of Note IV, "Sur la géométrographie," pages 517–548 of ROUCHÉ et COMBEROUSSE, *Traité de géométrie*, tome I, 8e éd., Paris, Gauthier-Villars, 1912; to (3) E. HAENTZSCHEL, "De l'exactitude des constructions géométriques," *L'Enseignement Mathématique*, tome 9, 1907,

[1] *Die geometrischen Constructionen ausgeführt mittelst der Geraden Linie und eines festen Kreises*, Berlin, 1833, § 19.

pp. 45–51; and to (4) recent volumes of *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, which had a department, under the editorship of K. Hagge, devoted to the discussion of problems in geometrography.

For extension of the considerations to space, see Lemoine's papers: (1) "La géométrographie dans l'espace," *Comptes rendus de l'Academie des Sciences*, Paris, vol. 131, 1900; (2) "Géométrographie dans l'espace ou stéréométrographie," *Comptes rendus de l'Association Française pour l'Avancement des Sciences*, 1900.

Among English writings the following may be noted: (1) J. S. MACKAY, "The Geometrography of Euclid's Problems," *Proceedings of the Edinburgh Mathematical Society*, vol. 12, 1894, pp. 2–16; (2) J. L. COOLIDGE, *A Treatise on the Circle and the Sphere*, Oxford, Clarendon Press, 1916, pp. 166–179; (3) H. P. HUDSON, *Ruler and Compasses*, London, Longmans, 1916, pp. 112–117; (4) C. E. YOUNGMAN, "On Two Constructions for the Regular 17-Side," *Mathematical Questions and Solutions from "The Educational Times,"* new series, vol. 10, 1906, pp. 55–56; and (5) R. F. MUIRHEAD, "Constructions with Straight-edge and Dividers," *Mathematical Gazette*, London, 1905, vol. 3, pp. 209–211.

Other systems of measurement are described in A. GRÜTTNER, *Die Grundlagen der Geometrographie*, Leipzig, Quelle und Meyer, 1912; in K. ROHN und E. PAPPERITZ, *Lehrbuch der darstellenden Geometrie*, Band I, 4. Auflage, Leipzig, Veit, 1913, pp. 486–493 and 501–502; and in *Encyklopädie der mathematischen Wissenschaften*, Band III1, Heft 4, Leipzig, Teubner, 1910, pp. 528–531.

The issue of the MONTHLY for February, 1896, contained a portrait of Lemoine and a biographical sketch by D. E. Smith. Reference is made to the influence which he exerted in the realm of music through his celebrated soirées, "La Trompette." (*Cf.* L. AUGÉ DE LASSUS, *La Trompette. Un demi-siècle de musique de chambre.* Paris, Delagrave, 1911.)

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Mr. D. R. BELCHER has been appointed instructor in mathematics at Adelbert College, Western Reserve University.

At Syracuse University, Associate Professor F. F. DECKER has been promoted to a professorship of mathematics, and Dr. J. L. JONES has been promoted to an assistant professorship.

Following the death of President P. W. MCREYNOLDS, of Defiance College, Defiance, O., Professor A. G. CARIS, of the department of mathematics, was made acting president of the College.

At the University of Maine, Mr. M. F. JORDAN and Mr. Q. STAUFFER have been appointed instructors in mathematics; the department of mathematics has united with the military department in offering courses in navigation.

Large numbers of teachers of mathematics in secondary schools and colleges have enlisted in the military service since the opening of the present academic year. It has been the plan of the editors of the MONTHLY to record as many of these enlistments as possible, especially the members of the Association. The editor of the "Notes and News" columns of the MONTHLY welcomes all notices sent him concerning the military service of any member of the Association, or other mathematicians.

Mr. GILBERT THAYER, a charter member of the Association, has been commissioned first lieutenant in the aviation section of the Signal Officers' Reserve Corps, in California. At the University of Maine, Associate Professor L. J. REED has been granted leave of absence to serve as statistician to the War Trade Board in Washington. Dr. J. M. STETSON, instructor in mathematics at Adelbert College of Western Reserve University, has resigned to enter the national army. Dr. A. H. NORTON, professor of mathematics and vice-president of Elmira College, N. Y., and president of the Syracuse section of the Mathematics Teachers of the Middle States and Maryland, has been granted leave of absence by Elmira College and sailed for France in December to engage in Y. M. C. A. work. Professor W. MARSHALL, of Purdue University, is serving under the Food Administration as chief statistician of the International Sugar Committee with headquarters in New York City.

The following mathematical papers were presented at the October meeting of the Cambridge Philosophical Society: "The convergence of certain multiple series," by G. H. HARDY; "Bessel functions of large order," by G. N. WATSON; "A particular case of a theorem of Dirichlet;" "Ramanujan's empirical expansion of modular functions," by L. J. MORDELL; "Extension of Abel's theorem and its converses," by A. KIENAST.

Parts 3 and 4 of Vol. 16, *Proceedings of the London Mathematical Society*, recently issued, contain the following: "Bessel functions and Kapteyn series" (continued from part 2), by G. N. WATSON; "On non-absolutely convergent, not necessarily continuous, integrals" and "On multiple integration by parts and the second theorem of the mean," by W. H. YOUNG; "On the classification of the integrals of linear partial differential equations of the first order," by M. J. M. HILL; "A problem on Diophantine approximation," by H. T. J. NORTON; "A symmetrical relation between Legendre's functions with parameters cosh α, sinh α," by F. J. W. WHIPPLE; "On two theorems of combinatory analysis and some allied identities," by L. J. ROGERS. At the regular meeting of the London Mathematical Society held on November 1 under the presidency of Professor H. M. MACDONALD, the following papers were presented: "Tetrahedra in relation to spheres and quadrics," by J. H. GRACE; "The continuation of the hypergeometric series," by M. J. M. HILL; "Restricted Fourier series and the convergence of power series," by W. H. YOUNG; "Invariants and covariants of linear homogeneous differential equations," by E. B. STOUFFER; "The simultaneous system of two quaternary quadratic forms," by H. W. TURNBULL.

Among the multitude of scientific papers presented at the meetings of the Paris Academy of Science during the past six months are the following on mathematical subjects: "Surfaces such that the Laplace equation of the network formed by the lines of curvature is integrable," by C. GUICHARD; "A generalization of Taylor's series," by G. D. BIRKHOFF; "Binary indeterminate conjugate forms remaining invariant under a group of linear substitutions," by G. JULIA; "Fourier-Bessel transcendentals with several variables," by M. AKIMOFF; "The convergence of conjugate trigonometrical series," by J. PRIWALOFF; "The development in a continued fraction of a quadratic irrational," by M. AMSLER; "The measurement of linear ensembles," by M. LEAU; "The continued fraction of Stephen Smith" and "Some properties of binary indefinite quadratic forms," by G. HUMBERT; "The classification of the transcendental points of the inverse of integral or meromorphic functions," by G. REMOUNDOS; "Orthogonal surfaces," by H. DUPORT; "The pisiform surface," by P. HUMBERT; "The notion of neighborhood in abstract ensembles," by M. FRÉCHET; "A new method in the numerical evaluation of the coefficients of a series," by M. PETROVITCH; "A new method in the summation of trigonometrical series," by M. ANGELESCO; "The theory of trigonometrical series," by W. H. YOUNG; "Abelian functions," by G. SCORZA; "The integration of certain systems of differential equations," by E. GOURSAT; "The triple cyclic systems of Steiner," by S. BAYS; "A functional equation and spherical unicursal curves," by W. DETAUNENBERG.

The volumes of the *Proceedings of the Edinburgh Mathematical Society* are issued annually in two parts, the first in May and the second in November. The recently completed volume 35, issued by the aid of a grant from the Carnegie Trustees, contains the following papers extending over 106 pages: By H. DATTA, (1) "On the theory of continued fractions"; (2) "On the failure of Heilemann's theorem"; by L. R. FORD, (1) "On a class of continued fractions"; (2) "A geometrical proof of a theorem of Hurwitz"; (3) "The effect of a rise in prices upon the amount of small money used"; by G. B. JEFFERY, "Transformations of axes for Whittaker's solution of Laplace's equation"; by W. P. MILNE, (1) "The apolar locus of two tetrads of points"; (2) "The co-apolars of a cubic curve"; by G. D. TAYLOR, "Geometrical illustrations of cyclant substitutions"; by E. T. WHITTAKER, "On the latent roots of compound determinants and Brill's determinants"; by A. W. YOUNG, "On the computation of a Lagrangian interpolation."

In addition to the *Proceedings* the Edinburgh Mathematical Society has published also 21 numbers of *Mathematical Notes, a Review of Elementary Mathematics and Science*. The last number (pages 239–254) was published in December, 1916.

Professor F. R. MOULTON, University of Chicago, is Secretary of Section A (Mathematics and Astronomy) of the American Association for the Advancement of Science. His absence from the holiday meetings at Chicago was due to his official duties at Pittsburgh.

The twenty-fourth annual meeting of the American Mathematical Society was held in New York on December 27–28, 1917. There were eleven papers on the printed program, the authors representing seven colleges and universities.

The ninth regular meeting of the American Mathematical Society at Chicago, being the fortieth meeting of the Chicago Section, was held at the University of Chicago on December 28–29, 1917, in conjunction with the third annual meeting of the Mathematical Association of America. There were eighteen papers on the printed program, the authors representing nine colleges and universities. Two additional papers were also read.

Former Captain STUART C. GODFREY, Corps of Engineers, West Point, New York, is now Major Godfrey and is located at Fort Leavenworth, Kan.

A booklet on the "History of Limits and Fluxions in England during the Eighteenth Century," by Professor FLORIAN CAJORI, is now in process of publication by the Oxford University Press, having been delayed since 1915 by the exigencies of the war.

A series of public addresses in the state of Montana on the nations concerned in the great war was given during the Red Cross campaign by members of the faculty of the University of Montana, and the proceeds were devoted to the Red Cross. The address on Scandinavia was given by Professor N. J. LENNES, head of the department of mathematics.

The Central Association of Science and Mathematics Teachers was organized in Chicago in 1902. The membership has steadily grown from 324 in 1908 to 973 in 1917. The last annual meeting was held in Columbus, Ohio. An important feature of the program was the report of the Committee on Mathematical Requirements by the chairman, Mr. J. A. FOBERG, of Crane Junior College, Chicago. Mr. Foberg is the representative of the Central Association on the National Committee of the Mathematical Association of America.

The secretary has had specific inquiries for the following back numbers of the AMERICAN MATHEMATICAL MONTHLY. It will confer a favor upon various subscribers who are seeking to complete their files of this journal if any who have these issues for sale will inform the secretary of the ASSOCIATION, giving the price desired, including the necessary postage. The secretary's office will be glad to effect the exchange in such cases if the price is satisfactory to the subscribers. (The numbers called for by the University of Wisconsin in the last issue of the MONTHLY have been supplied.)

Vol. I, 1894—Jan., Feb., Mar., June, and July.
Vol. II, 1895—June and Nov.
Vol. III, 1896 (at least all issues except Jan., Feb., and June).
Vol. IV, 1897—Jan., Feb., Apr., June, and July.
Vol. V, 1898—March and May.
Vol. VI, 1899—May, September, and November.
Vol. VII, 1900—April and May.

Vol. VIII, 1901—November.
Vol. IX, 1902—May and June–July.
Vol. XI, 1909—April.
Vol. XIV, 1907—Jan. and Oct.
Vol. XIX, 1912—June and July.
Vol. XX, 1913—Oct.

Professor W. D. MACMILLAN, who spent the autumn in Texas, has now returned to his work at the University of Chicago in the department of astronomy. Professors E. J. WILCZYNSKI and H. E. SLAUGHT are out of residence during the winter quarter, and Professor L. E. DICKSON has returned from his sojourn at the University of California.

NOTES ON THE THIRD ANNUAL MEETING OF THE ASSOCIATION.

By H. E. SLAUGHT, University of Chicago.

The attendance at the third annual meeting was most gratifying, in view of war conditions. The total number present at the various sessions was 119, including 93 members. Members from a great distance were warmly welcomed, including Professors Cajori from Colorado, Dodd and Ettlinger from Texas, Huntington, Phillips, and Ransom from Massachusetts, Archibald from Rhode Island, and Kingston from Winnipeg.

The joint dinner on Thursday evening with the Chicago Section of the Society was attended by seventy-three persons and proved most enjoyable. The Quadrangle Club is admirably adapted for such occasions, providing ample room and complete seclusion, with full opportunity for social intercourse both before and after the dinner and, indeed, throughout the progress of the meetings. President Cajori presided and short talks were given by members representing various parts of the country. A matter of deep interest was the Secretary's roll-call of mathematicians in the national service, a manifestly incomplete list, which, however, was supplemented by numerous additions on the spot, and which it is desired to make as complete as possible for early publication in the MONTHLY.

As was the case at the New York meeting a year ago, there was much sustained enthusiasm over the election of officers which was closed at the business meeting late Friday afternoon. The contest was very close in some cases, especially in connection with the Council. The detailed figures will be given in the Secretary's official report. The result was as follows: E. V. Huntington, President; D. N. Lehmer and J. W. Young, Vice-Presidents; W. D. Cairns, Secretary-Treasurer; Florian Cajori, Elizabeth B. Cowley, G. A. Miller, and E. J. Wilczynski, members of the Council to serve till January 1920. The Secretary's figures will show that the total vote was not as large as one would suppose it should be. Probably

many have not yet fully realized that these are *real* elections and that each and every member has a voice in the determination of the affairs of the Association. Why should we not have at least 1,000 votes at these annual elections?

The program was pronounced on every hand to be excellent. It was the first occasion when voluntary papers were called for by the Committee, but it is understood that henceforth this is to be an integral part of all programs. This announcement is made thus early in order that all may be aware of the privilege and that many may take advantage of the coming months to prepare such papers for submission to the next program committee for approval long in advance of the next meeting. On this occasion the time was very short, too short, but evidently those who responded to the call had been preparing for emergencies—at any rate, the result was most gratifying. The programs of all the sectional meetings are evidently conducted entirely on this basis and with great success—indeed, the sectional meetings should prove to be a fruitful source for the discovery of good papers for the national programs.

The address of Professor Roever proved to be inspiring and the prolonged discussion which it aroused led to the expression of a desire to organize the college teachers of descriptive geometry as a special division of the Association—at present they have no organization and are laboring under the dilemma that they are recognized neither by the drawing departments nor by the mathematical departments. The feeling prevailed that we in the Association should welcome them into full mathematical fellowship and steps will soon be taken to invite them into the membership of the Association.

The financial report of the Secretary-Treasurer showed a safe balance on the 1917 business of the Association. Whether this will be maintained during 1918 depends upon the loyalty and support of the present membership. There will be the temptation—even the pressure—in the case of many to economize by dropping memberships in some of the scientific associations, but now is the time, of all times, when such organizations should be supported at all hazards—even at personal sacrifice on the part of individuals. Many such bodies are proposing to hold on the rolls of active membership all who are in the military or naval service of the country without their payment of dues if they so request. Our Council voted to put our Association among those who gladly perform this patriotic service, believing that our membership will back up this action by holding our numbers not only up to the present total but by actually increasing this total through still further accretions from the ranks of present non-members. It was surely gratifying to have twenty-two individuals and two institutions elected to membership at this meeting.

In spite of the heavy burden (over $400) for increased cost of paper for the MONTHLY during 1917, the Association carried on the work of its standing committees and provided the subvention agreed upon for the *Annals;* and we look forward with confidence to 1918 feeling assured that no backward step will be necessary on account of the lessening of individual support.

An important step forward was taken in connection with the amendment to the constitution providing for a further subdivision of the editorial responsibilities. Professor Carmichael now becomes editor-in-chief of the MONTHLY and a feeling of absolute confidence prevails throughout the Association that the MONTHLY could not be in better hands, and that we are entering upon an epoch of distinct advancement. Already, improvements and new departments are in process of development which will commend themselves to all readers of the journal. The personnel of the editorial committee remains unchanged except for the promotion of Professor Carmichael to the editorship. The business management of the MONTHLY remains as heretofore. In accordance with the amended by-laws, the associate editors are now appointed annually by the Council on the recommendation of the editorial committee. These appointments will appear in the official report of the Council actions. The Council held three sessions, ten members being present at each session.

In response to a call sent out previous to the Chicago meeting, some twenty-five Illinois members of the Association met on Thursday afternoon to consider the question of forming a section of the Association. The secretary's report of this meeting will appear in the February number of the MONTHLY.

An effort was made to compile a more serviceable index to the last volume of the MONTHLY than those in previous volumes, and it is hoped that this effort will appeal to our readers as worth while and justifying the added labor and expense involved. The work was done under the supervision of Professor Helen A. Merrill of the editorial staff.

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THIRD ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The third annual meeting of the Association was held at the University of Chicago on Thursday and Friday, December 27–28, 1917, in conjunction with the Chicago Section of the American Mathematical Society, which met on Friday and Saturday of that week. There were 119 in attendance at the various sessions, including the following 93 members of the Association and one institutional delegate:

L. D. AMES, University of Missouri.
R. C. ARCHIBALD, Brown University.

I. A. BARNETT, Chicago, Ill.
MRS. W. E. BECKWITH, College for Women, Western Reserve University.
G. A. BLISS, University of Chicago.
P. P. BOYD, University of Kentucky.
H. T. BURGESS, University of Wisconsin.
W. H. BUSSEY, University of Minnesota.

W. D. CAIRNS, Oberlin College.
FLORIAN CAJORI, Colorado College.
D. F. CAMPBELL, Armour Institute of Technology.
R. D. CARMICHAEL, University of Illinois.
W. E. CEDERBERG, Augustana College.
H. E. COBB, Lewis Institute.
L. M. COFFIN, Coe College.
C. E. COMSTOCK, Bradley Polytechnic Institute.
M. W. COULTRAP, North Western College.
A. R. CRATHORNE, University of Illinois.
G. H. CRESSE, University of Michigan.
D. R. CURTISS, Northwestern University.

ALFRED DAVIS, Parker High School, Chicago.
E. W. DAVIS, University of Nebraska.
L. E. DICKSON, University of Chicago.
E. L. DODD, University of Texas.
ARNOLD DRESDEN, University of Wisconsin.
OTTO DUNKEL, Washington University.

ARNOLD EMCH, University of Illinois.
L. C. EMMONS, Michigan Agricultural College.
H. T. ETTLINGER, University of Texas.

J. A. FOBERG, Crane Junior College, Chicago.
W. B. FORD, University of Michigan.
A. F. FRUMVELLER, Marquette University.

CORNELIUS GOUWENS, University of Iowa.
M. E. GRABER, Heidelberg University.

LAURENCE HADLEY, Earlham College.
W. A. HAMILTON, Beloit College.
HARRIS HANCOCK, University of Cincinnati.
J. O. HASSLER, Crane Junior College, Chicago.
E. R. HEDRICK, University of Missouri.
T. H. HILDEBRANDT, University of Michigan.
F. H. HODGE, Franklin College.
E. V. HUNTINGTON, Harvard University.